Adeles

Unit 13

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Algebras

Let K be a field, and R a ring (commutative with unit, as usual). If R is also a K-vector space we say that R is a K-algebra.

The definition of an algebra is in fact more general but the above works for us.

A typical example is the ring of polynomials $K[x_1, \ldots, x_n]$.

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Definition 1 (Adeles)

An adele of a function field F/K is a map $\alpha : \mathbb{P} \to F$ satisfying $v_{\mathfrak{p}}(\alpha(\mathfrak{p})) \geq 0$ almost always.

This somewhat weird-looking definition turns out to encode a proper balance of the "global" and "local" properties of the function field.

Remarks.

- We will write $\alpha_{\mathfrak{p}}$ for $\alpha(\mathfrak{p})$.
- The set $\mathbb{A} = \mathbb{A}_{F/K}$ of adeles of F/K is a ring with point-wise addition and multiplication.

$$(\alpha\beta)(\mathfrak{p}) = \alpha(\mathfrak{p})\beta(\mathfrak{p}),$$
$$(\alpha + \beta)(\mathfrak{p}) = \alpha(\mathfrak{p}) + \beta(\mathfrak{p}).$$

Remarks.

- We identify F as a subfield of $\mathbb{A}$ via
$$x \mapsto [x] \ \text{ s.t. } [x]_{\mathfrak{p}} = x \quad \forall \mathfrak{p} \in \mathbb{P}.$$
- $\mathbb{A}_{F/K}$ is an F-algebra: For $x \in F$
$$x\alpha = [x]\alpha.$$
More explicitly,
$$(x\alpha)_{\mathfrak{p}} = x \cdot \alpha_{\mathfrak{p}}.$$
- We extend the valuation $\upsilon_{\mathfrak{p}}$ from F to $\mathbb{A}$ by
$$\upsilon_{\mathfrak{p}}(\alpha) = \upsilon_{\mathfrak{p}}(\alpha_{\mathfrak{p}}).$$

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Recall that for $\mathfrak{a} \in \mathcal{D}$ we defined the Riemann-Roch space

$$\mathcal{L}(\mathfrak{a}) = \{x \in \mathsf{F} \mid v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0 \text{ for all } \mathfrak{p} \in \mathbb{P}\}.$$

Definition 2

For a divisor $\mathfrak{a}$ let

$$\Lambda(\mathfrak{a}) = \{\alpha \in \mathbb{A} \mid v_{\mathfrak{p}}(\alpha) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0 \text{ for all } \mathfrak{p} \in \mathbb{P}\}.$$

Proving the following claim is left as an exercise.

Claim 3

- $\Lambda(\mathfrak{a})$ is a K-vector space, a subspace of $\mathbb{A}$.
- $\mathcal{L}(\mathfrak{a}) = \Lambda(\mathfrak{a}) \cap \mathsf{F}$.

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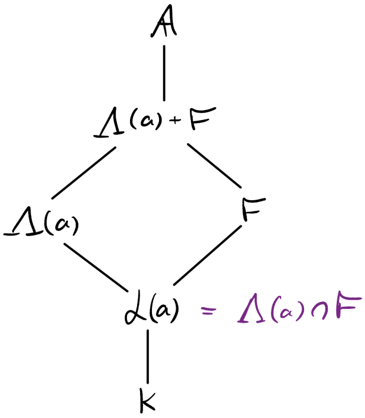

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Claim 4

For divisors $\mathfrak{a}, \mathfrak{b}$ and $x \in \mathrm{F}$

- $\mathfrak{a} \leq \mathfrak{b} \implies \Lambda(\mathfrak{a}) \subseteq \Lambda(\mathfrak{b})$.
- $\Lambda(\mathfrak{a}) \cap \Lambda(\mathfrak{b}) = \Lambda(\min(\mathfrak{a}, \mathfrak{b}))$.
- $\Lambda(\mathfrak{a}) + \Lambda(\mathfrak{b}) = \Lambda(\max(\mathfrak{a}, \mathfrak{b}))$.
- $x\Lambda(\mathfrak{a}) = \Lambda(\mathfrak{a} - (x))$.

The proof is similar to that for Riemann-Roch spaces and is left as an exercise.

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Recall that for divisors $\mathfrak{a} \le \mathfrak{b}$,

$$\dim_K \left(\mathcal{L}(\mathfrak{b}) \big/ \mathcal{L}(\mathfrak{a})\right) \le \deg \mathfrak{b} - \deg \mathfrak{a}.$$

Moreover, if $S \subset \mathbb{P}$ is the prime divisors appearing in at least one of $\mathfrak{a}, \mathfrak{b}$, then

$$\dim_K \left(\mathcal{L}(\mathfrak{b}, S) \big/ \mathcal{L}(\mathfrak{a}, S)\right) = \deg \mathfrak{b} - \deg \mathfrak{a}.$$

Lemma 5

For divisors $\mathfrak{a} \le \mathfrak{b}$,

$$\dim_K \left(\Lambda(\mathfrak{b}) \big/ \Lambda(\mathfrak{a})\right) = \deg \mathfrak{b} - \deg \mathfrak{a}.$$

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Proof.

Let $S \subset \mathbb{P}$ be the prime divisors appearing in at least one of $\mathfrak{a}, \mathfrak{b}$. We proved that

$$\dim_K \left(\mathcal{L}(\mathfrak{b}, S) \big/ \mathcal{L}(\mathfrak{a}, S) \right) = \deg \mathfrak{b} - \deg \mathfrak{a}.$$

So it suffices to establish the isomorphism (as K-vector spaces)

$$\Lambda(\mathfrak{b}) \big/ \Lambda(\mathfrak{a}) \cong \mathcal{L}(\mathfrak{b}, S) \big/ \mathcal{L}(\mathfrak{a}, S).$$

Let $T : \mathrm{F} \to \mathbb{A}$ be the map $x \mapsto T(x)$ that is given by

$$T(x)_{\mathfrak{p}} = \begin{cases} x & \text{if } \mathfrak{p} \in S; \\ 0 & \text{otherwise.} \end{cases}$$

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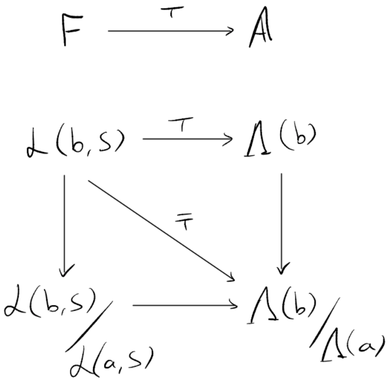
Proof.

$$T(x)_{\mathfrak{p}} = \begin{cases} x & \text{if } \mathfrak{p} \in S; \\ 0 & \text{otherwise.} \end{cases}$$

T is a K-linear map that maps $\mathcal{L}(\mathfrak{a}, S)$ into $\Lambda(\mathfrak{a})$ and $\mathcal{L}(\mathfrak{b}, S)$ into $\Lambda(\mathfrak{b})$. Indeed,

$$\begin{aligned} x \in \mathcal{L}(\mathfrak{a}, S) \quad &\Longrightarrow \quad \forall \mathfrak{p} \in S \quad v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0 \\ &\Longrightarrow \quad \forall \mathfrak{p} \in \mathbb{P} \quad v_{\mathfrak{p}}(T(x)) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0 \\ &\Longrightarrow \quad T(x) \in \Lambda(\mathfrak{a}). \end{aligned}$$

T composed with the projection map $\Lambda(\mathfrak{b}) \to \Lambda(\mathfrak{b}) \big/ \Lambda(\mathfrak{a})$ gives rise to a map $\bar{T}$.



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Proof.

We first prove that $\ker \bar{T} = \mathcal{L}(\mathfrak{a}, S)$. Indeed,

$$\begin{aligned}\ker \bar{T} &= \{x \in \mathcal{L}(\mathfrak{b}, S) \mid T(x) \in \Lambda(\mathfrak{a})\} \\ &= \{x \in \mathcal{L}(\mathfrak{b}, S) \mid \forall \mathfrak{p} \in \mathbb{P} \quad v_{\mathfrak{p}}(T(x)) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0\}.\end{aligned}$$

For $\mathfrak{p} \in \mathbb{P} \setminus S$, $v_{\mathfrak{p}}(T(x)) = v_{\mathfrak{p}}(0) = \infty$ and $v_{\mathfrak{p}}(\mathfrak{a}) = 0$, and so

$$\begin{aligned}\ker \bar{T} &= \{x \in \mathcal{L}(\mathfrak{b}, S) \mid \forall \mathfrak{p} \in S \quad v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0\} \\ &= \mathcal{L}(\mathfrak{a}, S).\end{aligned}$$

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Proof.

Next, we show that $\bar{T}$ is onto.

Take $\beta \in \Lambda(\mathfrak{b})$. By WAT $\exists x \in F$ s.t.

$$\forall \mathfrak{p} \in S \quad v_\mathfrak{p}(T(x) - \beta_\mathfrak{p}) = v_\mathfrak{p}(x - \beta_\mathfrak{p}) \geq -v_\mathfrak{p}(\mathfrak{a}),$$

As for $\mathfrak{p} \in \mathbb{P} \setminus S$,

$$v_\mathfrak{p}(T(x) - \beta_\mathfrak{p}) = v_\mathfrak{p}(\beta_\mathfrak{p}) \geq -v_\mathfrak{p}(\mathfrak{b}) = 0 = -v_\mathfrak{p}(\mathfrak{a}).$$

So, $T(x) - \beta \in \Lambda(\mathfrak{a})$. Hence, $\bar{T}(x) = \beta + \Lambda(\mathfrak{a})$.

We just need to now that $x \in \mathcal{L}(\mathfrak{b}, S)$. Indeed, for $\mathfrak{p} \in S$,

$$v_\mathfrak{p}(x) \geq \min(v_\mathfrak{p}(x - \beta_\mathfrak{p}), v_\mathfrak{p}(\beta_\mathfrak{p})) \geq \min(-v_\mathfrak{p}(\mathfrak{a}), -v_\mathfrak{p}(\mathfrak{b})) = -v_\mathfrak{p}(\mathfrak{b}).$$

The proof then follows by the first isomorphism theorem for vector spaces. □

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Recap. For divisors $\mathfrak{a} \leq \mathfrak{b}$ we just proved that

$$\dim_K \left(\Lambda(\mathfrak{b}) \middle/ \Lambda(\mathfrak{a})\right) = \deg \mathfrak{b} - \deg \mathfrak{a}.$$

Lemma 6

For divisors $\mathfrak{a} \leq \mathfrak{b}$,

$$\dim_K \left(\Lambda(\mathfrak{b}) + F \middle/ \Lambda(\mathfrak{a}) + F\right) = (\deg \mathfrak{b} - \dim \mathfrak{b}) - (\deg \mathfrak{a} - \dim \mathfrak{a}).$$

For the proof of Lemma 6 we recall the third isomorphism theorem for vector spaces.

Let $U \subseteq V \subseteq W$ be K-vector spaces. Then, V/U is a subspace of W/U, and

$$(W/U) \middle/ (V/U) \cong W \middle/ V.$$

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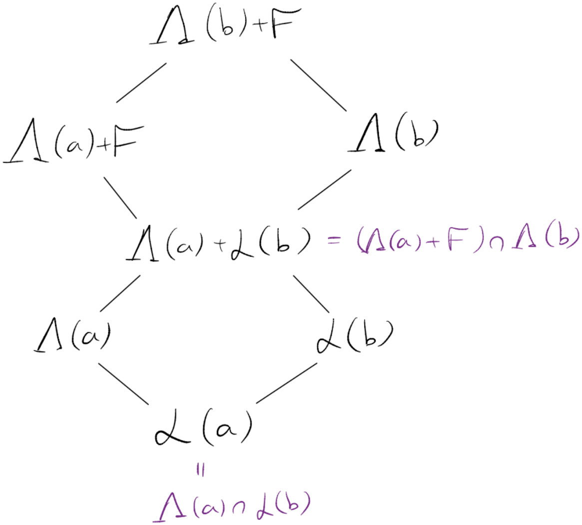

Proof.

By Claim 3,

$$\mathcal{L}(\mathfrak{a}) = \Lambda(\mathfrak{a}) \cap F.$$

But since $\mathcal{L}(\mathfrak{a}) \subseteq \mathcal{L}(\mathfrak{b})$,

$$\begin{aligned} \mathcal{L}(\mathfrak{a}) &= \mathcal{L}(\mathfrak{a}) \cap \mathcal{L}(\mathfrak{b}) \\ &= (\Lambda(\mathfrak{a}) \cap F) \cap \mathcal{L}(\mathfrak{b}) \\ &= \Lambda(\mathfrak{a}) \cap (F \cap \mathcal{L}(\mathfrak{b})) \\ &= \Lambda(\mathfrak{a}) \cap \mathcal{L}(\mathfrak{b}). \end{aligned}$$

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Proof.

We turn to prove that

$$(\Lambda(\mathfrak{a}) + \mathsf{F}) \cap \Lambda(\mathfrak{b}) = \Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b}).$$

The $\supseteq$ direction is easy.

Now, take an element in the LHS and write it as $\alpha + x$ with $\alpha \in \Lambda(\mathfrak{a})$ and $x \in \mathsf{F}$. But $\Lambda(\mathfrak{a}) \subseteq \Lambda(\mathfrak{b})$ and so

$$\alpha + x \in \Lambda(\mathfrak{b}) \quad \Longrightarrow \quad x \in \Lambda(\mathfrak{b}).$$

But $x \in \mathsf{F}$ and so

$$x \in \Lambda(\mathfrak{b}) \cap \mathsf{F} = \mathcal{L}(\mathfrak{b}).$$

Thus,

$$\alpha + x \in \Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b}).$$

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Proof.

By the diagram and by the second isomorphism theorem,

$$(\Lambda(\mathfrak{b}) + F) \Big/ (\Lambda(\mathfrak{a}) + F) \cong \Lambda(\mathfrak{b}) \Big/ (\Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b})),$$

and by the third and second isomorphism theorems,

$$\begin{aligned}\Lambda(\mathfrak{b}) \Big/ (\Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b})) &\cong \Lambda(\mathfrak{b}) \Big/ \Lambda(\mathfrak{a}) \Bigg/ (\Lambda(\mathfrak{a}) + \mathcal{L}(\mathfrak{b})) \Big/ \Lambda(\mathfrak{a}) \\ &\cong \Lambda(\mathfrak{b}) \Big/ \Lambda(\mathfrak{a}) \Bigg/ \mathcal{L}(\mathfrak{b}) \Big/ \mathcal{L}(\mathfrak{a})\end{aligned}$$

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Proof.

So,

$$\begin{aligned}\dim_K \left(\Lambda(\mathfrak{b}) + F \middle/ \Lambda(\mathfrak{a}) + F\right) &= \dim_K \Lambda(\mathfrak{b}) \middle/ \Lambda(\mathfrak{a}) - \dim_K \mathcal{L}(\mathfrak{b}) \middle/ \mathcal{L}(\mathfrak{a}) \\ &= \deg \mathfrak{b} - \deg \mathfrak{a} - (\dim \mathfrak{b} - \dim \mathfrak{a}) \\ &= (\deg \mathfrak{b} - \dim \mathfrak{b}) - (\deg \mathfrak{a} - \dim \mathfrak{a}).\end{aligned}$$

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Corollary 7

$$\deg \mathfrak{a} - \dim \mathfrak{a} = g - 1 \quad \Longrightarrow \quad \Lambda(\mathfrak{a}) + \mathsf{F} = \mathbb{A}.$$

Proof.

Take any divisor $\mathfrak{b} \geq \mathfrak{a}$. By monotonicity,

$$g - 1 \geq \deg \mathfrak{b} - \dim \mathfrak{b} \geq \deg \mathfrak{a} - \dim \mathfrak{a} = g - 1.$$

Lemma 6 then implies that

$$\Lambda(\mathfrak{b}) + \mathsf{F} = \Lambda(\mathfrak{a}) + \mathsf{F}.$$

Now, take $\alpha \in \mathbb{A}$ and choose $\mathfrak{b} \geq \mathfrak{a}$ s.t.

$$\forall \mathfrak{p} \in \mathbb{P} \quad v_{\mathfrak{p}}(\mathfrak{b}) + v_{\mathfrak{p}}(\alpha) \geq 0.$$

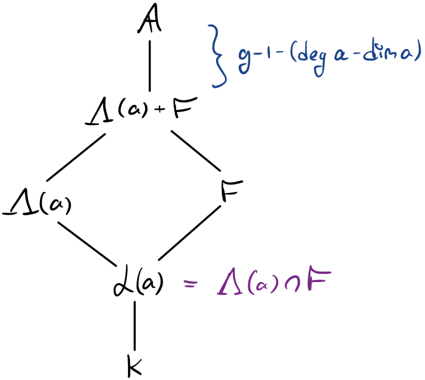
Thus,

$$\alpha \in \Lambda(\mathfrak{b}) \subseteq \Lambda(\mathfrak{a}) + \mathsf{F}.$$

Corollary 8

For every divisor $\mathfrak{a}$,

$$\dim_K \mathbb{A} \Big/ (\Lambda(\mathfrak{a}) + F) = g - 1 - (\deg \mathfrak{a} - \dim \mathfrak{a}).$$



Proof.

Take any divisor $\mathfrak{b} \geq \mathfrak{a}$ with

$$\deg \mathfrak{b} - \dim \mathfrak{b} = g - 1.$$

By Corollary 7,

$$\Lambda(\mathfrak{b}) + \mathsf{F} = \mathbb{A}.$$

Lemma 6 then yields

$$\begin{aligned}\dim_{\mathsf{K}} \mathbb{A} \Big/ (\Lambda(\mathfrak{a}) + \mathsf{F}) &= \dim_{\mathsf{K}} \left(\Lambda(\mathfrak{b}) + \mathsf{F} \Big/ \Lambda(\mathfrak{a}) + \mathsf{F} \right) \\ &= (\deg \mathfrak{b} - \dim \mathfrak{b}) - (\deg \mathfrak{a} - \dim \mathfrak{a}) \\ &= g - 1 - (\deg \mathfrak{a} - \dim \mathfrak{a}).\end{aligned}$$