Basic combinatorics of non-crossing partitions

Based on Nica-Speicher Chapter 9

Def. Let S be a finite totally ordered set.

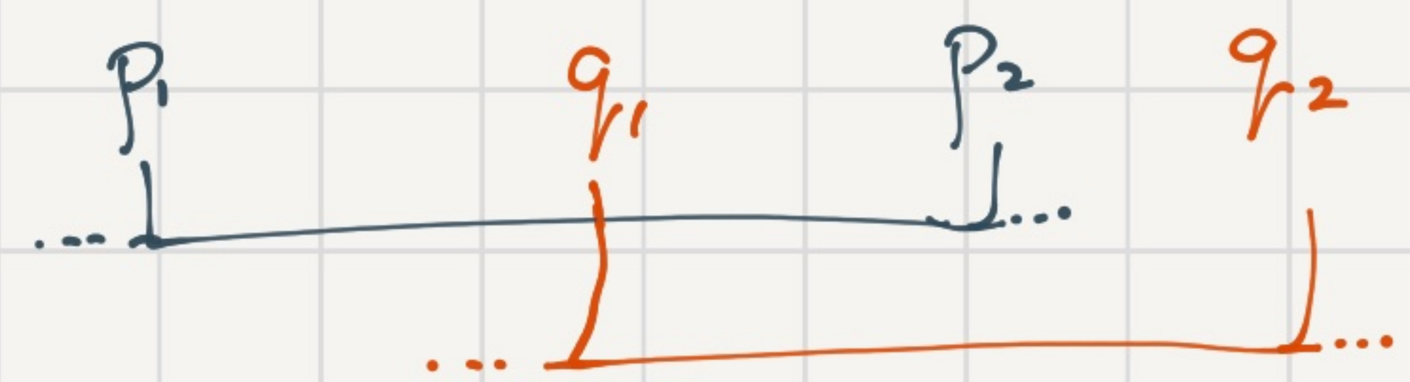
The set of all partitions of S is denoted as $\mathcal{P}(S)$.

When $S = [n]$ we write $\mathcal{P}(n)$.

We do not allow empty parts: $\emptyset \notin \pi$

A partition π of S is called <u>crossing</u> if

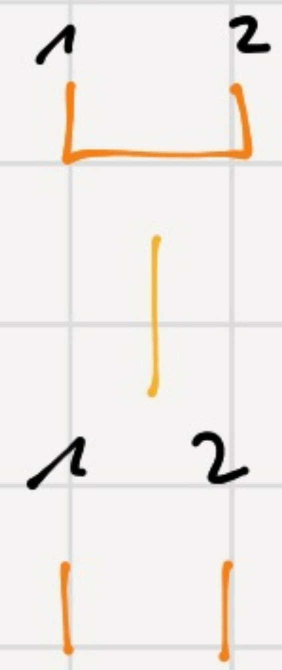
$\exists\, p_1 < q_1 < p_2 < q_2$ in S s.t. $p_1 \sim_\pi p_2 \not\sim_\pi q_1 \sim_\pi q_2$.



The set of all non-crossing partitions of S is denoted by

$NC(S)$ & $NC(n)$ for $S = [n]$.

NC(2):



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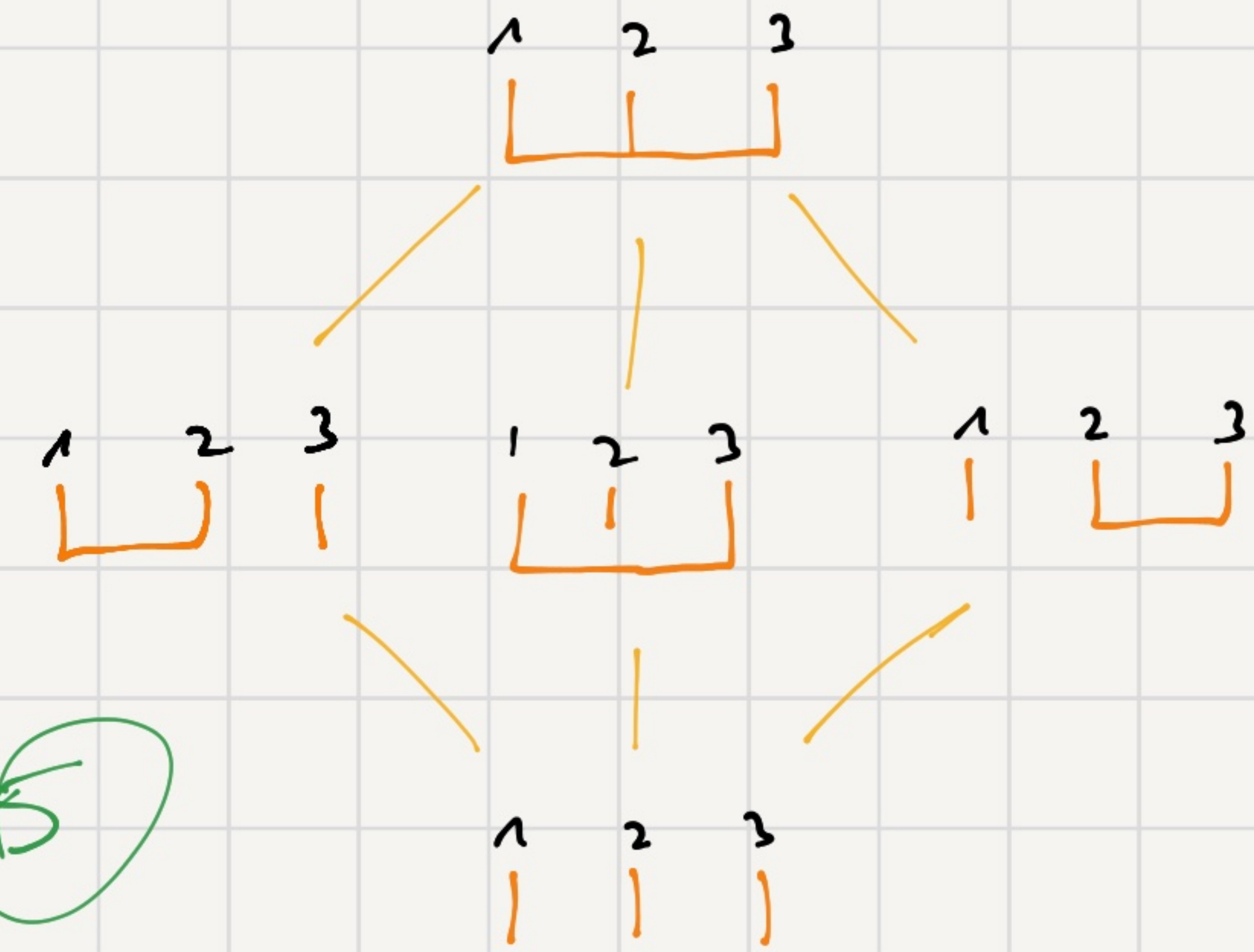
NC(3):

= P(3)



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NC(4):

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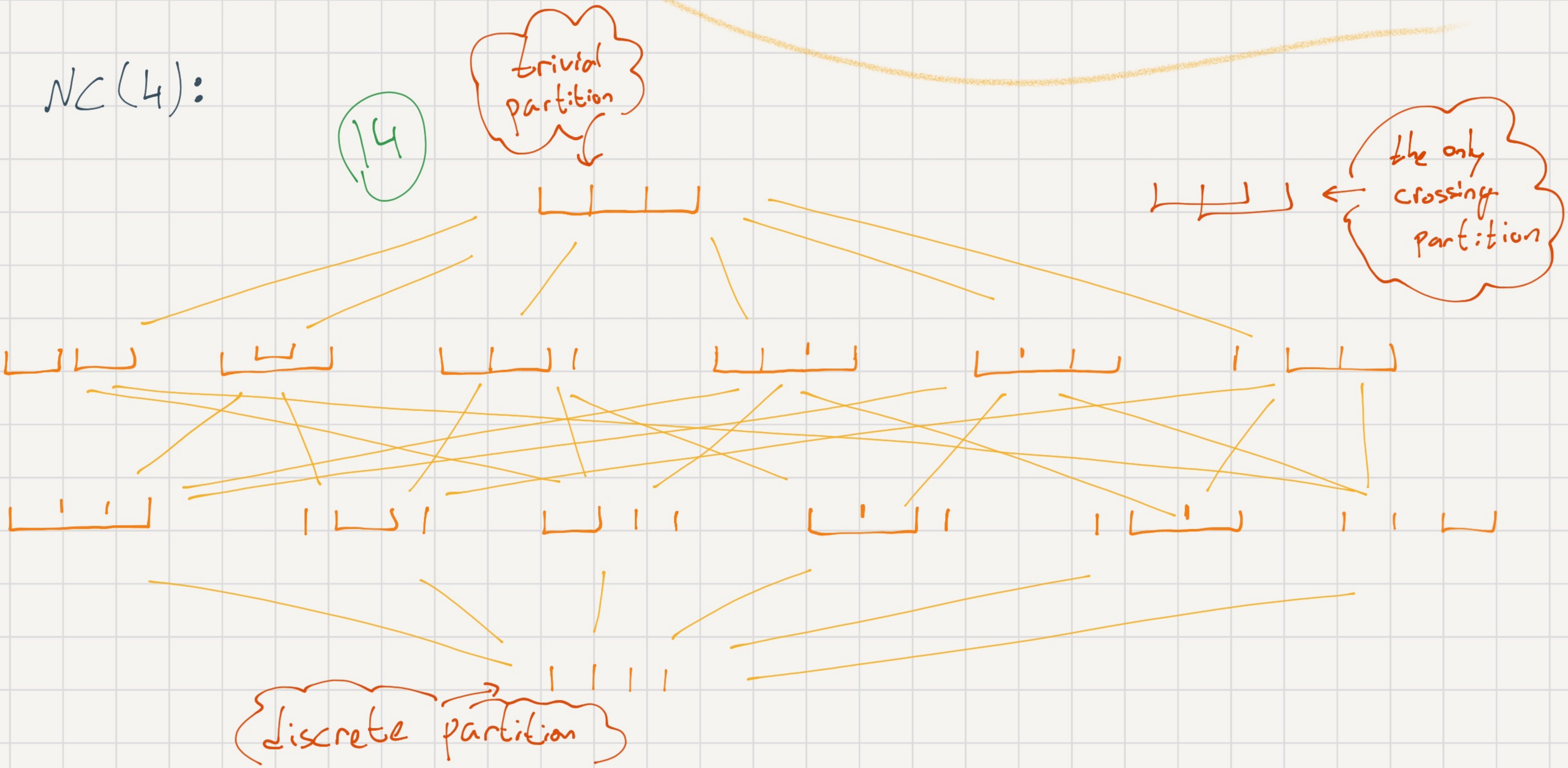


Obs. $\pi \in P(n)$ is non-crossing $\iff$ $\exists V \in \pi$ which is an interval $(V = \{r, r+1, \dots, s\})$ & $\pi \setminus V$ is non-crossing.

Notation. Let S be a totally ordered set & $\emptyset \neq W \subseteq S$, with the order induced from S. For $\pi \in NC(S)$ we denote by $\pi|_W$ the restriction of π to W:

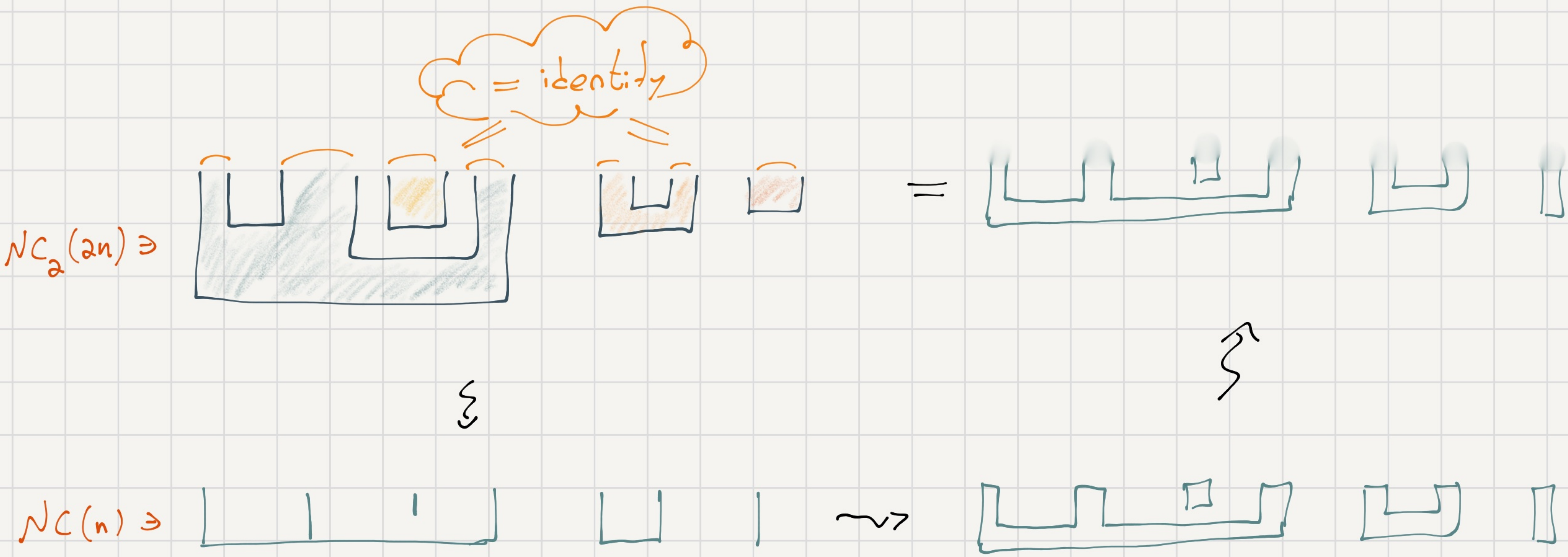
$$\pi|_W = \{V \cap W \mid V \in \pi\} \setminus \{\emptyset\}$$

Theorem $|NC(n)| = C_n$.

pf. Can be done by showing that the recurrence relation for $|NC(n)|$ (and the initial condition) match that of Catalan.

We'll see a different proof: $NC(n) \cong NC_2(2n)$

pf by picture.



Exercise. formalize & prove (it is somewhat insightful!)

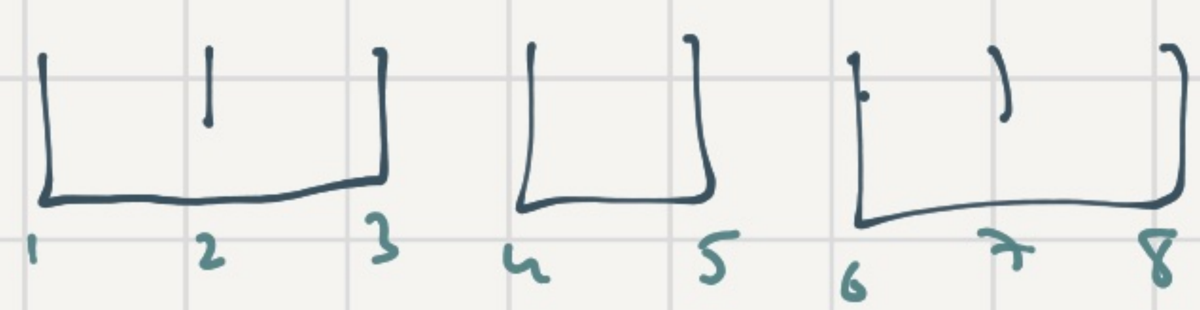
The lattice structure of $NC(n)$

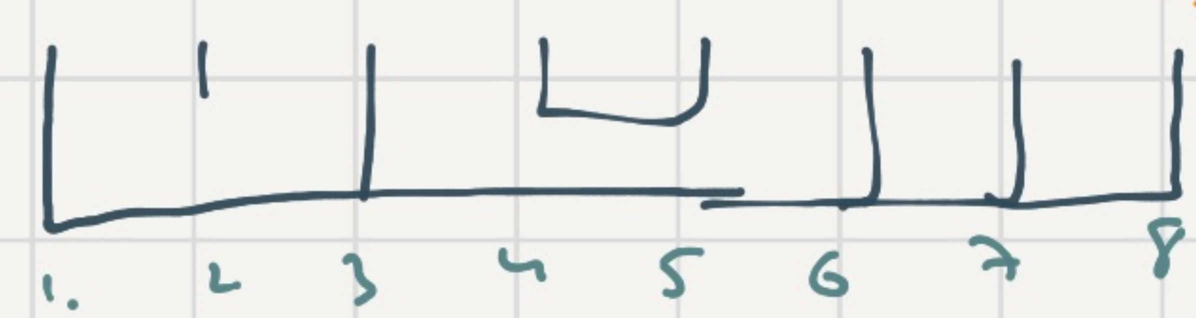
reflexive: $x \le x$
transitive: $x \le y$ & $y \le z \Rightarrow x \le z$
anti-symmetric: $x \le y$ & $y \le x \Rightarrow x = y$

$NC(n)$ is a partially ordered set (poset):

Def. Let $\pi, \sigma \in NC(n)$. We write $\pi \le \sigma$ if each block of π is is contained in some block of σ.

π is a refinement of σ

E.g. 

$\le$ 

$\{\{1,3\}, \{2\}, \{4,5\}, \{6,8\}, \{7\}\}$ $\quad$ $\{\{1,3,6,7,8\}, \{2\}, \{4,5\}\}$

This is called the reversed refinement order.

The maximal element is the one-block partition.

The minimum element is the partitions to singletons.

The "trivial partition"

The "discrete partition"

$NC(n)$ is not just a poset but is in fact a <u>lattice</u>.

<u>Def.</u> Let P be a <u>finite</u> poset.

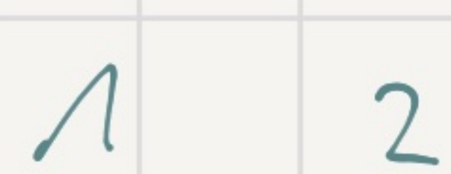
$\tau_0 \leq u \ \forall u \in U$

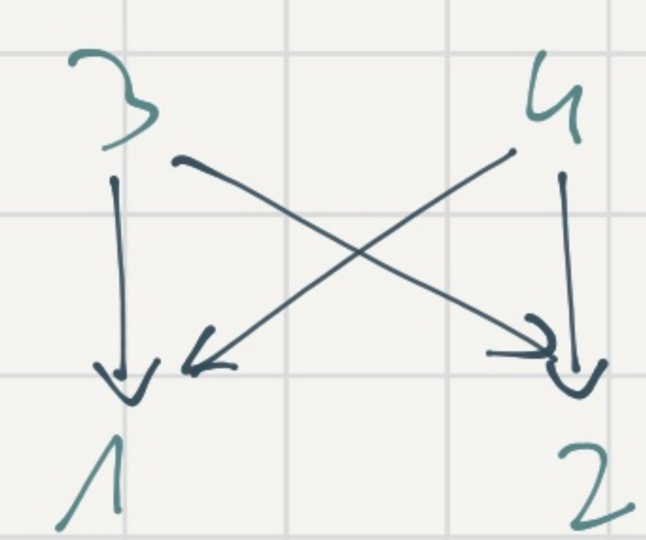
* For $\pi, \sigma \in P$ if the set $U = \{ \tau \in P \mid \tau \geq \pi \ \& \ \tau \geq \sigma \}$ is non-empty & has a minimum $\tau_0 \in U$ then τ_0 is called the <u>join</u> of π & σ and is denoted as $\pi \vee \sigma$

* For $\pi, \sigma \in P$ if the set $L = \{\rho \in P \mid \rho \leq \pi \ \& \ \rho \leq \sigma\}$ is non-empty & has a maximum $\rho_0 \in L$ then ρ_0 is called the meet $\rho_0 \geq u \ \forall u \in L$ of π & σ and is denoted as $\pi \wedge \sigma$.

* The poset P is said to be a lattice if $\forall \pi, \sigma \in P$ $\exists$ a join $\pi \vee \sigma$ & a meet $\pi \wedge \sigma$.

Nonexamples.





Examples

just some finite set

$\ast$ $P=$ power set of X w.r.t $\subseteq$: For $A, B \subseteq X$

$A \leq B \iff A \subseteq B$. Then

$$A \vee B = \text{minimum of } U = \{C \subseteq X \mid A \subseteq C \ \& \ B \subseteq C\} = A \cup B$$

$$A \wedge B = \text{maximum of } L = \{C \subseteq X \mid C \subseteq A \ \& \ C \subseteq B\} = A \cap B$$

$\ast$ $P = \mathbb{N}$ with $n \leq m \iff n \mid m$.

$$n \vee m = \text{minimum of } U = \{k \in \mathbb{N} \mid n \mid k \ \& \ m \mid k\} = \operatorname{lcm}(n, m).$$

$$n \wedge m = \text{maximum of } L = \{k \in \mathbb{N} \mid k \mid n \ \& \ k \mid m\} = \gcd(n, m).$$

Remarks

* Let P be a lattice. By a simple induction, every $\pi_1, \dots, \pi_k$ have a join (smallest common upper bound) denoted $\pi_1 \vee \dots \vee \pi_k$ and a meet $\pi_1 \wedge \dots \wedge \pi_k$ (largest common lower bound).

$\vee$ is indeed associative: $(\pi \vee \sigma) \vee \tau = \pi \vee (\sigma \vee \tau)$

and so is $\wedge$

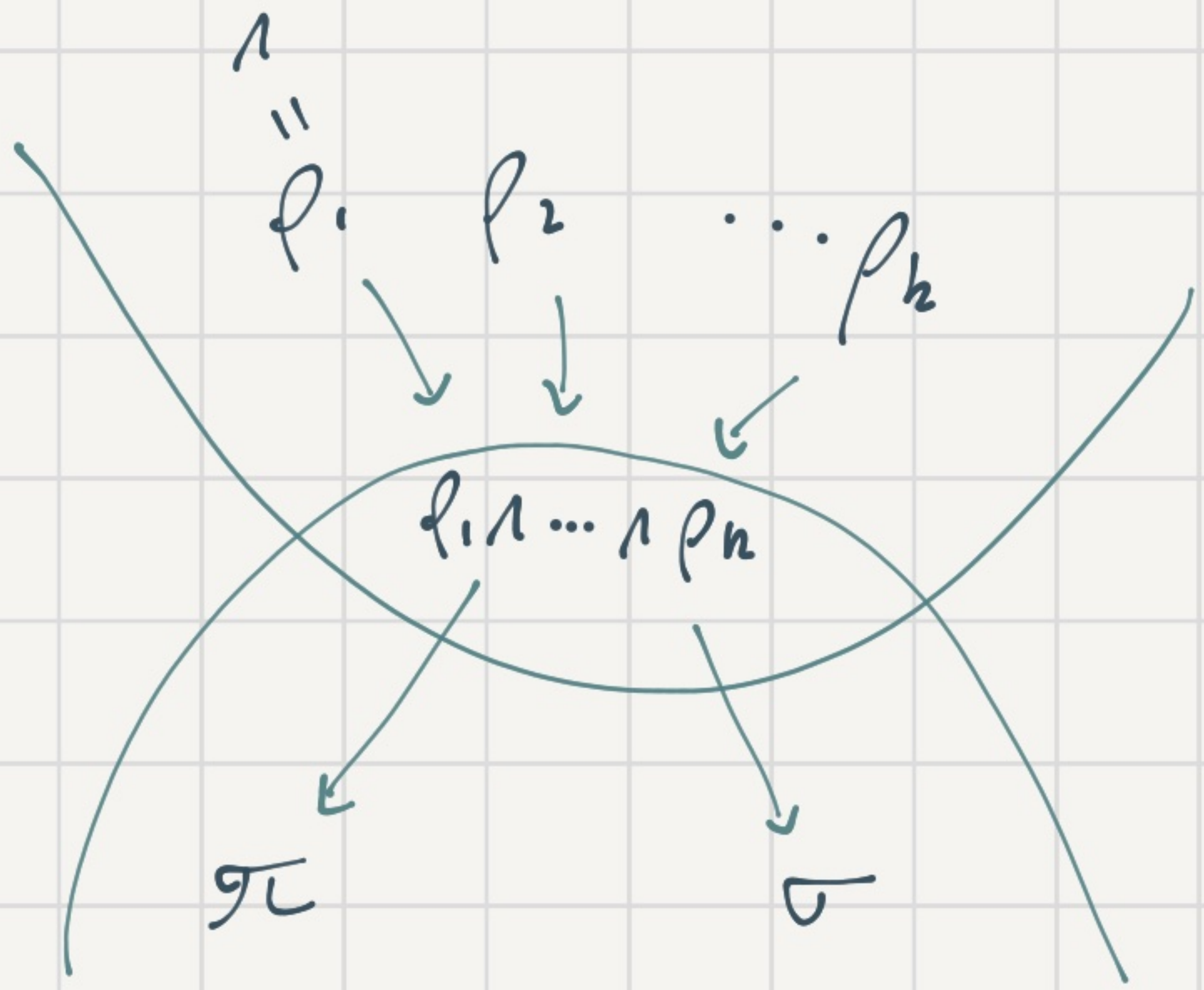
* In particular, P has a maximum denoted 1_P & a minimum element denoted 0_P: $\forall \pi \in P \quad 0_P \leq \pi \leq 1_P$.

* Let P be a finite poset with a maximal element 1_P. Then for P to be a lattice it suffices that every two elements have a meet.

Indeed, take π, σ. Then, $U = \{\tau \in P \mid \tau \geq \pi \ \& \ \tau \geq \sigma\} \ni 1_P$ and so is non-empty & finite, so $U = \{\rho_1, \dots, \rho_k\}$. Thus,

$$\pi \vee \sigma = \rho_1 \wedge \cdots \wedge \rho_k$$

verify!
(see figure)



E.g. $A \cup B = \bigcap_{A, B \subseteq C} C$

* Similarly, if a finite poset P has a minimum 0_P & every $\pi, \sigma \in P$ have a join then P is a lattice.

Proposition. The partial order by reversed refinement induces a lattice structure on $NC(n)$.

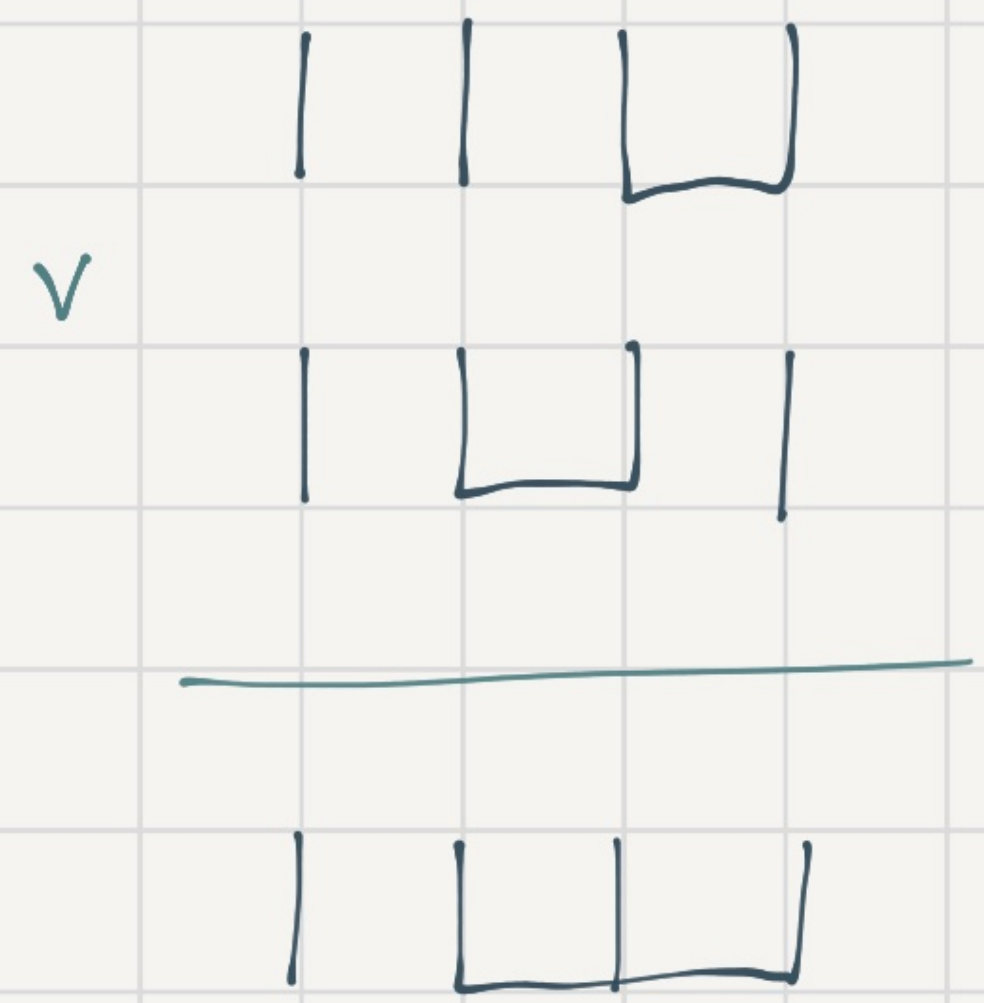
pf. By the above, and since $1_{NC(n)} = \sqcup\!\sqcup\!\sqcup\cdots\sqcup$ is a maximal element in $NC(n)$ it suffices to show that every $\pi, \sigma \in NC(n)$ have a meet $\pi \wedge \sigma$.

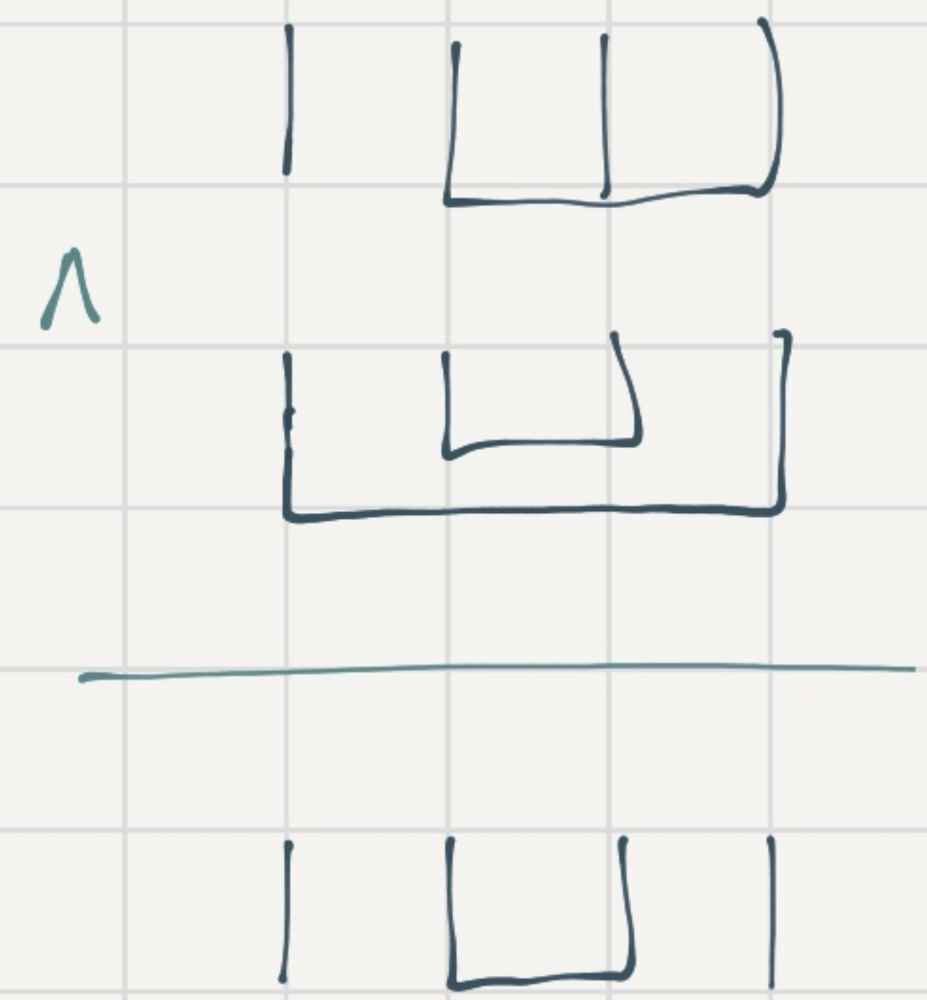
Indeed, if $\pi = \{V_1, \dots, V_r\}$, $\sigma = \{W_1, \dots, W_s\}$ then

$$\{V_i \cap W_j \mid i \in [r],\ j \in [s] \text{ s.t. } V_i \cap W_j \neq \emptyset\}$$

is a partition in $NC(n)$ which is smaller than π & σ and is the largest partition in $NC(n)$ having this property.

Example.





Remark. The partial order by reversed refinement can also be be considered on $P(n)$, and turns $P(n)$ into a lattice.

all partitions of $[n]$

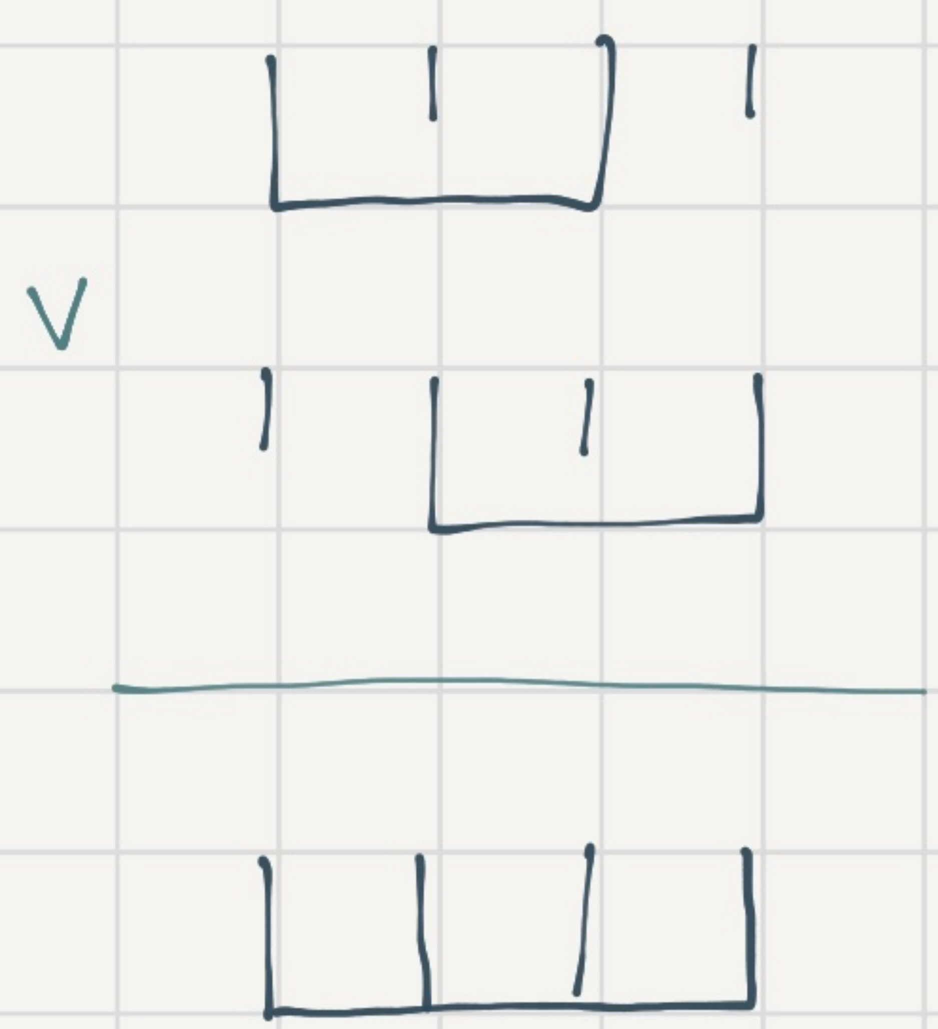
Taking intersections doesn't introduce crossings

The meet $\pi \wedge \sigma$ in this lattice is the same of $NC(n)$.

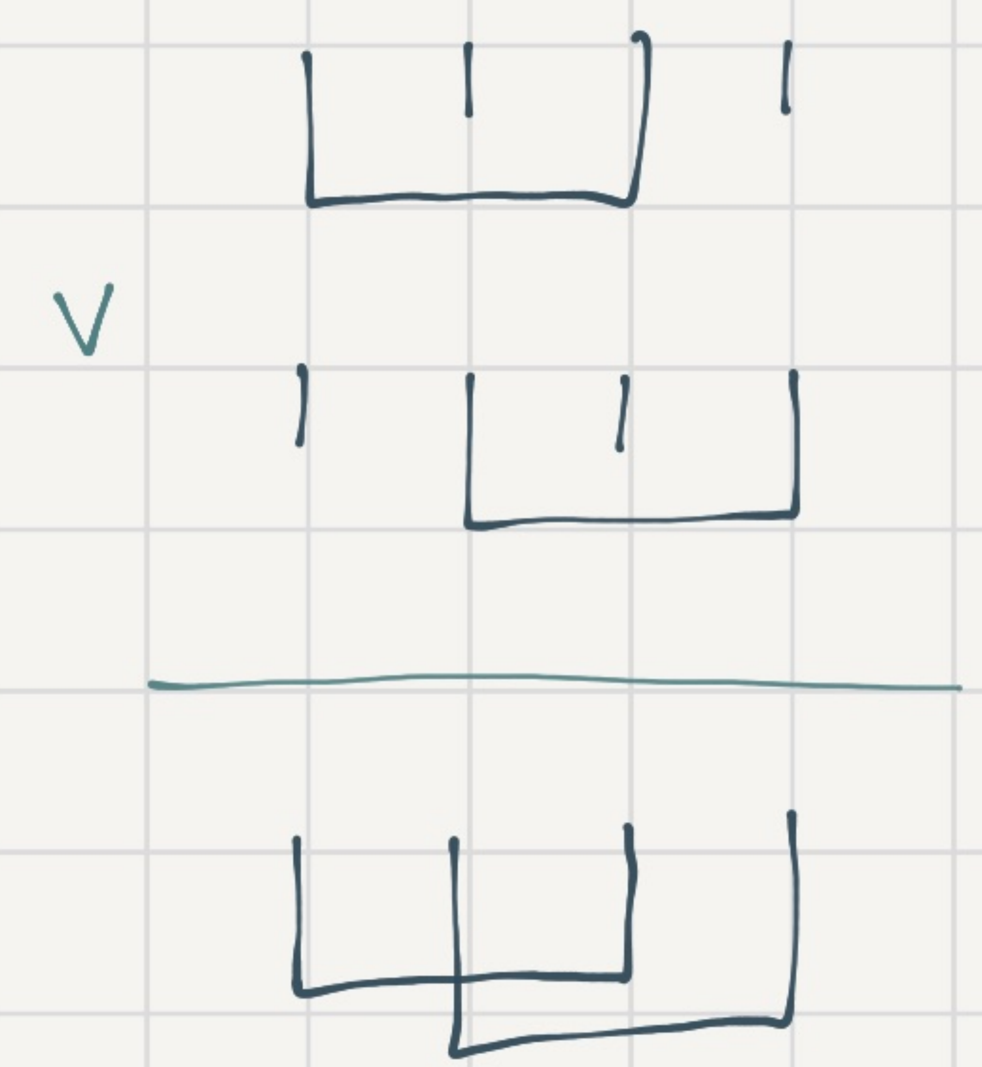
The join is different.

E.g.

$NC(4)$



$P(4)$



Some basic definitions on posets

Def. Two posets $(P, \leq_P)$, $(Q, \leq_Q)$ are isomorphic if $\exists\, \varphi: P \to Q$ bijection s.t. $x \leq_P y \iff \varphi(x) \leq_Q \varphi(y)$.

$\geq_Q$ anti-isomorphic

φ is an order-preserving bijection

Def. Let $(P, \leq_P)$ be a poset, $Q \subseteq P$. Q inherits the structure of the poset P: $\forall\, q_1, q_2 \in Q \quad q_1 \overset{\Delta}{\leq_Q} q_2 \iff q_1 \leq_P q_2$.

$(Q, \leq_Q)$ is called a subposet of $(P, \leq_P)$.

Def. Given a poset $(P, \leq)$ and $x \leq y$ in P, we define the interval $[x,y]$ by

typically considered as a subposet

$$[x,y] = \{ z \in P \mid x \leq z \leq y \}$$

Observation. If P is a lattice then so is $[x,y]$.

$\forall \sigma, \tau \in [x,y]$
$\sigma \wedge \tau \in [x,y]$ &
$\sigma \vee \tau \in [x,y]$

Def. Let $P_1, \ldots, P_n$ be posets. The direct product of $P_1, \ldots, P_n$, denoted $P_1 \times \cdots \times P_n$ is the poset given by

$$(\pi_1, \ldots, \pi_n) \leq (\sigma_1, \ldots \sigma_n) \iff \pi_i \leq \sigma_i \quad \forall i \in [n].$$

Observation. If $P_1, \ldots, P_n$ are lattices then so is $P_1 \times \cdots \times P_n$.

& $$(\pi_1, \ldots, \pi_n) \underset{(\wedge)}{\vee} (\sigma_1, \ldots, \sigma_n) = (\pi_1 \underset{(\wedge)}{\vee} \sigma_1, \ldots, \pi_n \underset{(\wedge)}{\vee} \sigma_n).$$

Kreweras

complement

Say $\pi \in NC(\{1,3,\dots,2n-1\})$, $\sigma \in NC(\{2,4,\dots,2n\})$. When does $\pi \cup \sigma \in NC(2n)$?

$\notin NC(2n)$

<u>Notation.</u> It will be more convenient to work with $\pi, \sigma \in NC(n)$ so given such we identify π with $\bar{\pi} \in NC(\{1,3,\dots,2n-1\})$ in the natural way & same for σ and $\bar{\sigma} \in NC(\{2,4,\dots,2n\})$ and define

$$\pi \bowtie \sigma \triangleq \bar{\pi} \cup \bar{\sigma}.$$

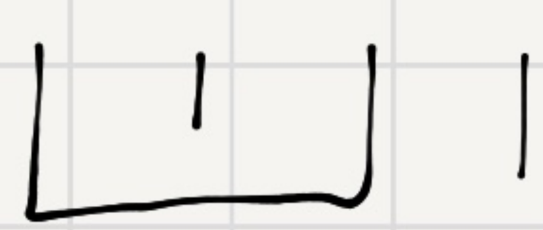
Def. Fix $\pi \in NC(n)$. Define

$$K_\pi = \{\sigma \in NC(n) \mid \pi \bowtie \sigma \in NC(2n)\}$$

Observation. $\forall \pi \quad 0 \in K_\pi \qquad (\Rightarrow K_\pi \neq \emptyset)$

Examples

* $\pi = 1_n \Longrightarrow K_\pi = \{0_n\}$:

* $\pi =$ 

$K_\pi = \Big\{$  $\Big\}$

Lemma. $\forall \pi \in NC(n)$ K_π is a lattice.

pf. K_π is automatically a subposet of $NC(n)$.

$\wedge$: If $\sigma \in K_\pi$ then $\sigma' \in K_\pi$ $\forall \sigma' \leq \sigma$. ← downwards closed

Thus, $\sigma, \tau \in K_\pi \Rightarrow \sigma \wedge \tau \in K_\pi$.

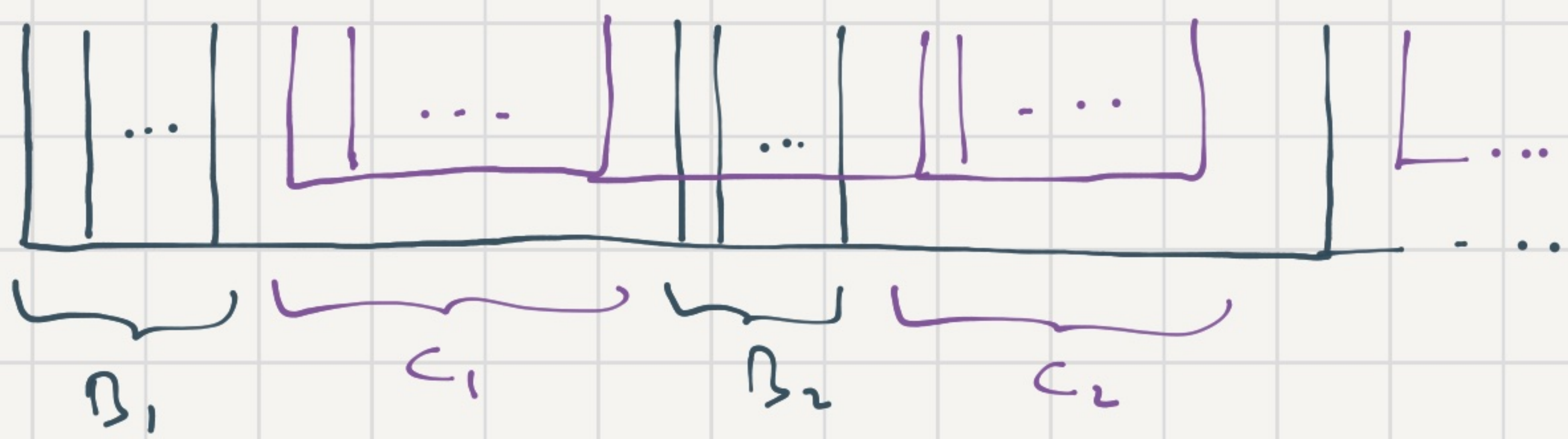
$\vee$: Recall that $\sigma \vee \tau = \wedge \underbrace{\{\lambda \in NC(n) \mid \lambda \geq \sigma \ \& \ \lambda \geq \tau\}}_{\mathcal{U}}$.

due to downward closedness

It is enough to show that $\mathcal{U} \cap K_\pi \neq \emptyset$.

Take $\lambda \in \mathcal{U}$. If $\lambda \in K_\pi$ (namely, $\pi \wedge \lambda$ doesn't have a crossing) then we're done. Otherwise, let $B \in \lambda$ be a block that crosses a block C of π

wlog. can also start with C



Write $B = B_1 \cup B_2 \cup \cdots \cup B_k$ ($k \geq 2$) $\quad C = C_1 \cup C_2 \cup \cdots \cup C_r$ ($r \geq 2$) s.t.

$\forall x \in B_1$ & $y \in C_1$ $x < y$

$$B_1 \leq C_1 \leq B_2 \leq C_2 \leq \cdots$$

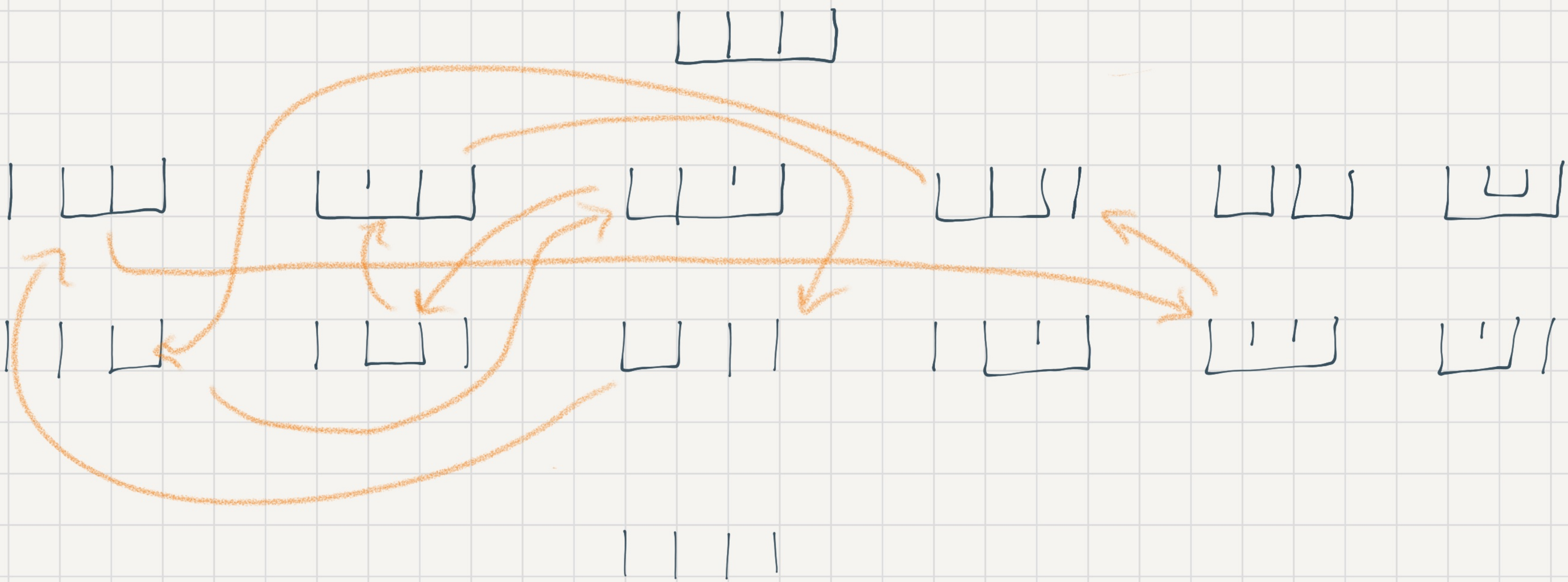
The refined partition

$$\lambda' \triangleq (\lambda \setminus \{B\}) \cup \{B_1, B \setminus B_1\}$$

still satisfies $\lambda' \geq \sigma$ & $\lambda' \geq \tau$ as both σ, τ do not cross π. If $\lambda' \in K_{\pi}$ we're done. Otherwise we repeat.

Each time the number of blocks of λ increases $\Rightarrow$ the process must terminate, and when it does we have $\lambda''' \in \mathcal{U} \cap K_{\pi}$. $\blacksquare$

Def. For $\pi \in NC(n)$ the *Kreweras complement* $K(\pi)$ is defined as the maximum element in the lattice K_π.



Lemma. $K_n : NC(n) \to NC(n)$ is an anti-homomorphism of posets: $\forall\, \pi \le \lambda \quad K_n(\lambda) \le K_n(\pi)$.

pf. $\lambda \wedge K_n(\lambda)$ is non-crossing & $\pi \le \lambda \implies$

$\pi \wedge K_n(\lambda)$ is also non-crossing, namely, $K_n(\lambda) \in K_\pi$

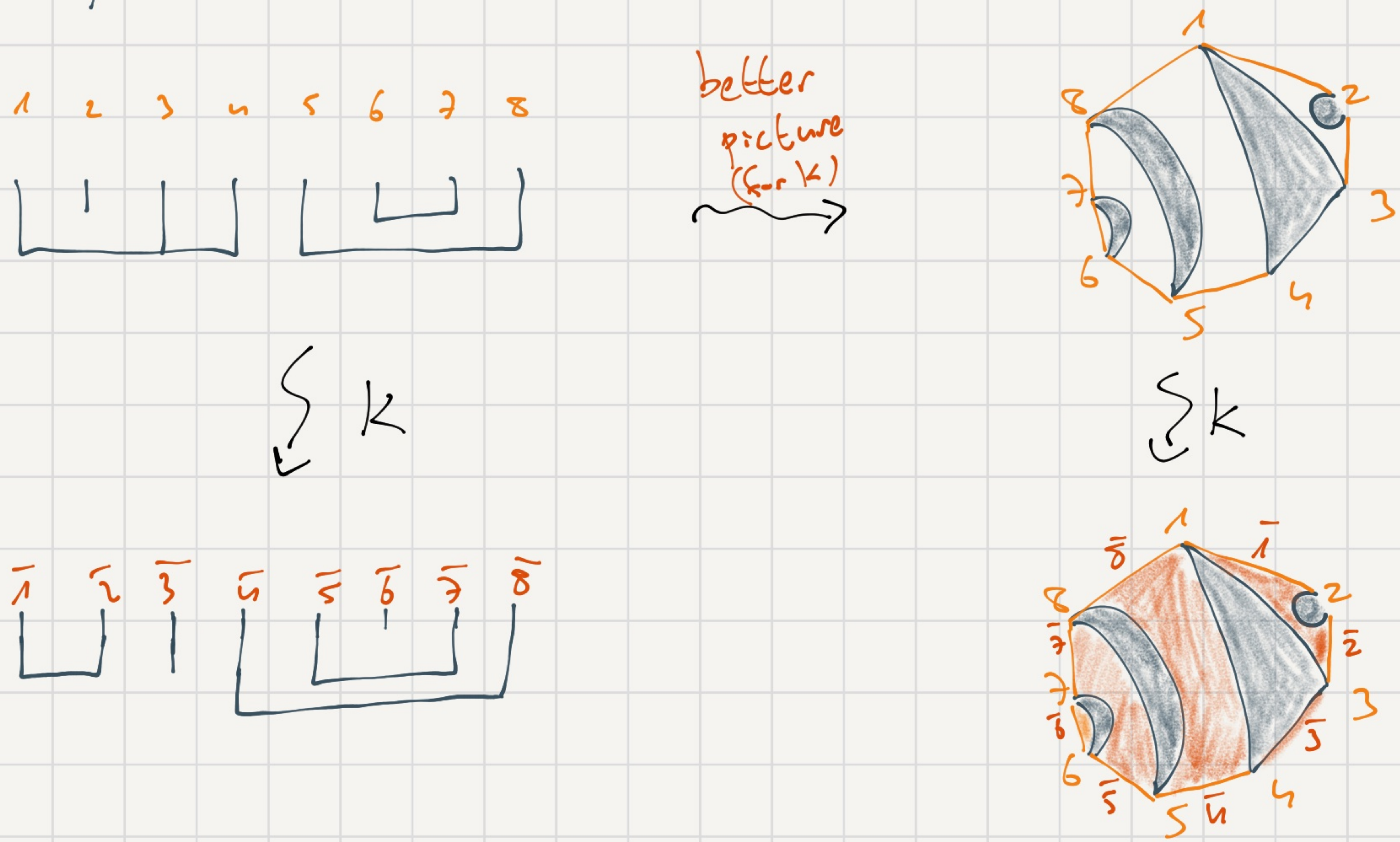
$\implies K_n(\lambda) \le K_n(\pi)$.

if we number from 0

Let c_n by the cyclic rotation to the right $(c_n(i) = i+1 \mod n)$

Lemma. $K_n^2 = c_n$

proof by picture.



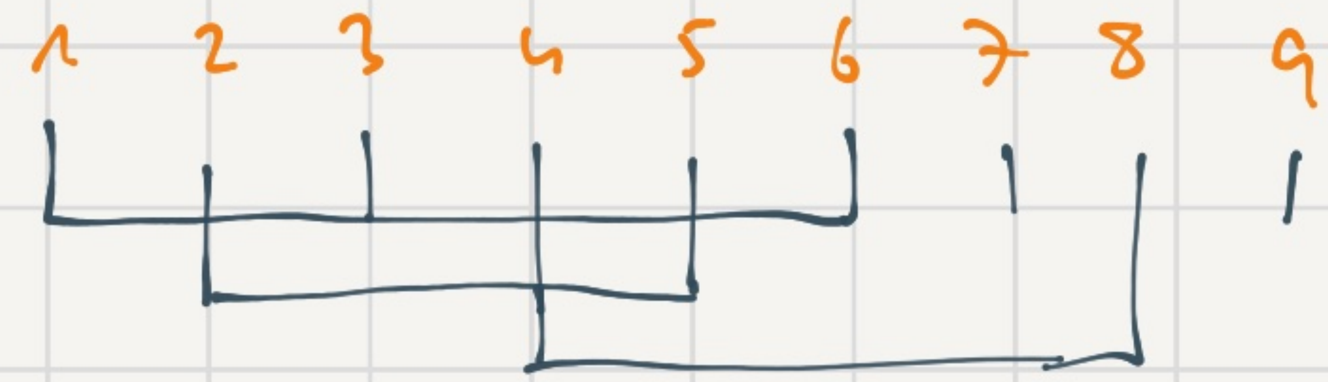
Corollary. K_n is an anti-automorphism of $NC(n)$

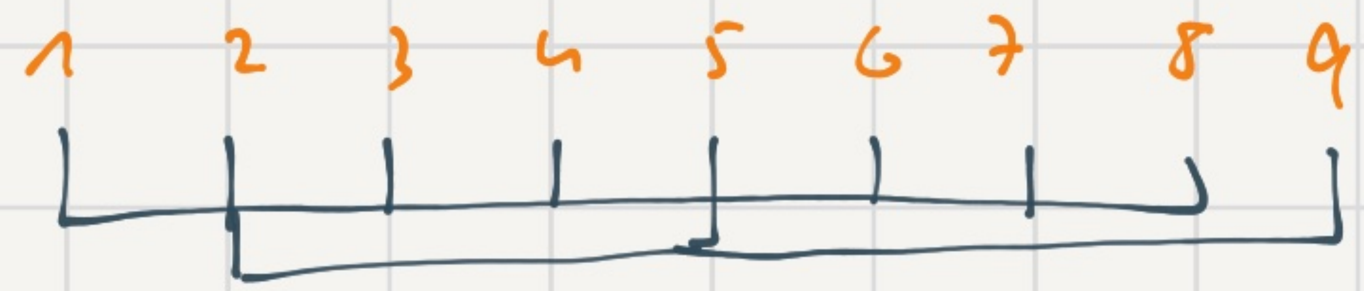
pf. $K_n^{2n} = C_n^n = Id \Longrightarrow K_n$ is a bijection. ∎

The factorization of intervals in NC

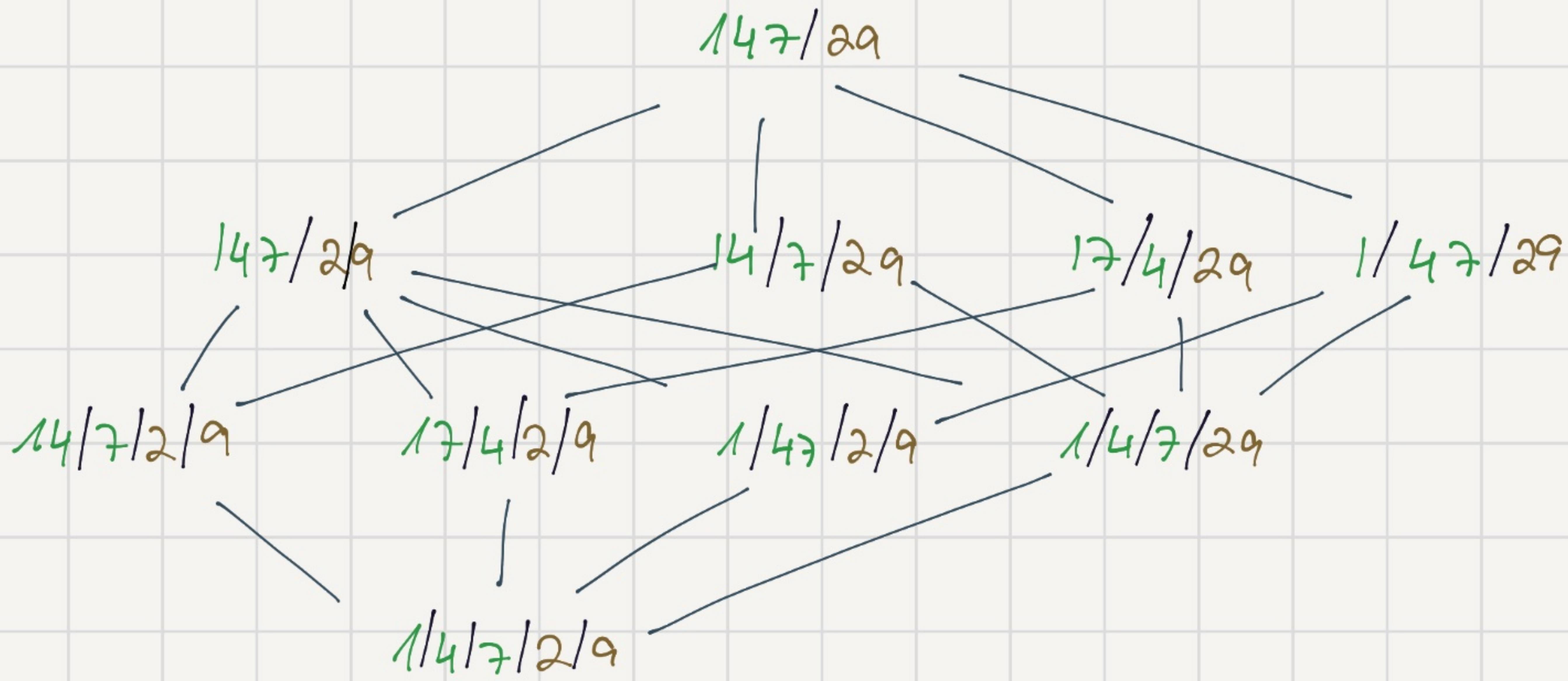
Consider the lattice of (all) partitions $P(9)$

$\sigma = 136/25/48/7/9$ $\qquad\qquad$ $\pi = 134678/259$





$[\sigma, \pi] =$

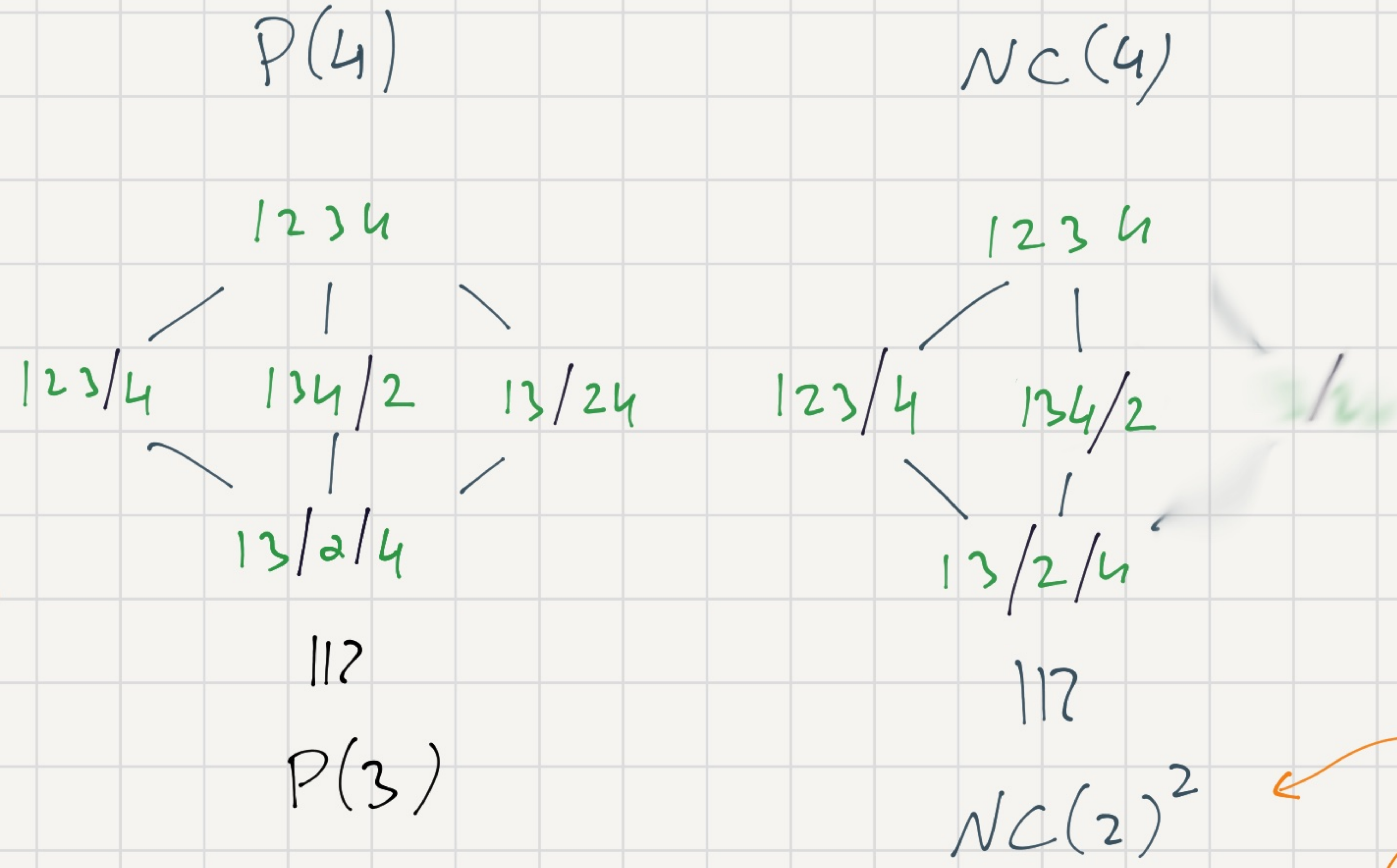


$\cong P(3) \times P(2)$

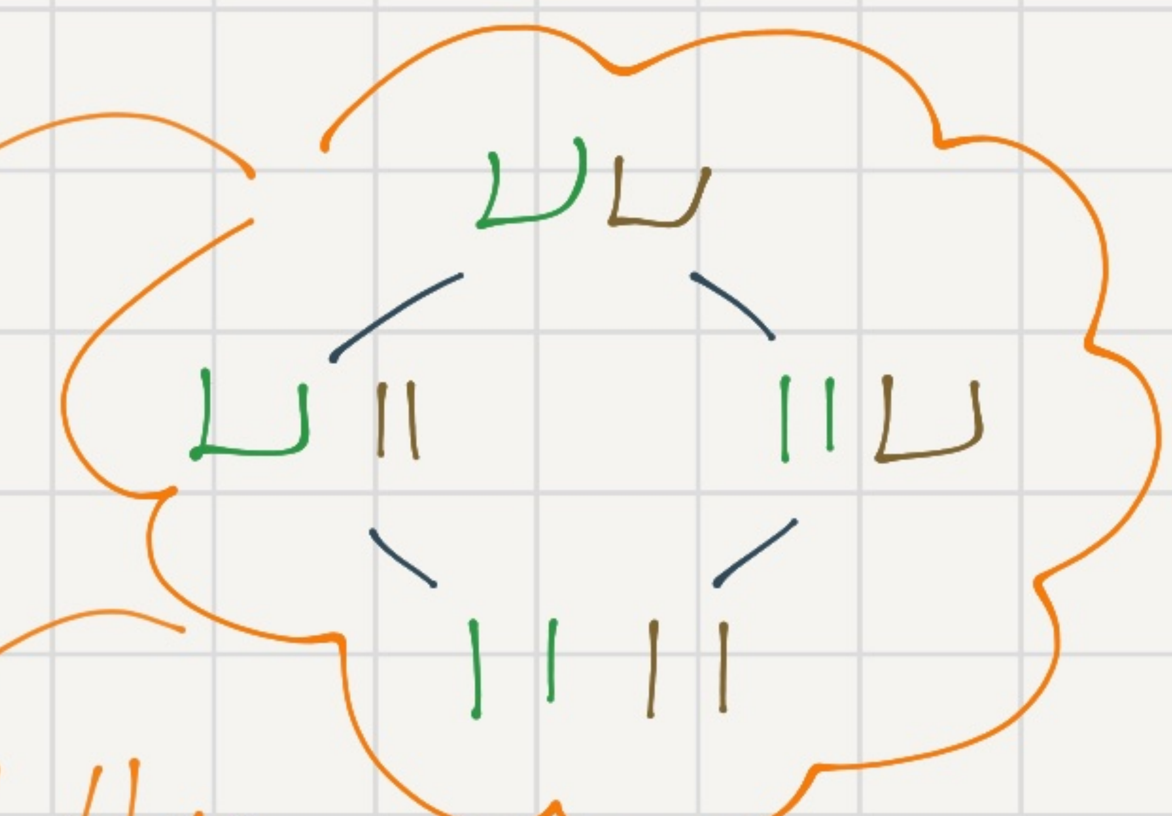
So any interval of $P(n)$ can be factored as follows: if π has b blocks & block i splits into n_i blocks in σ then

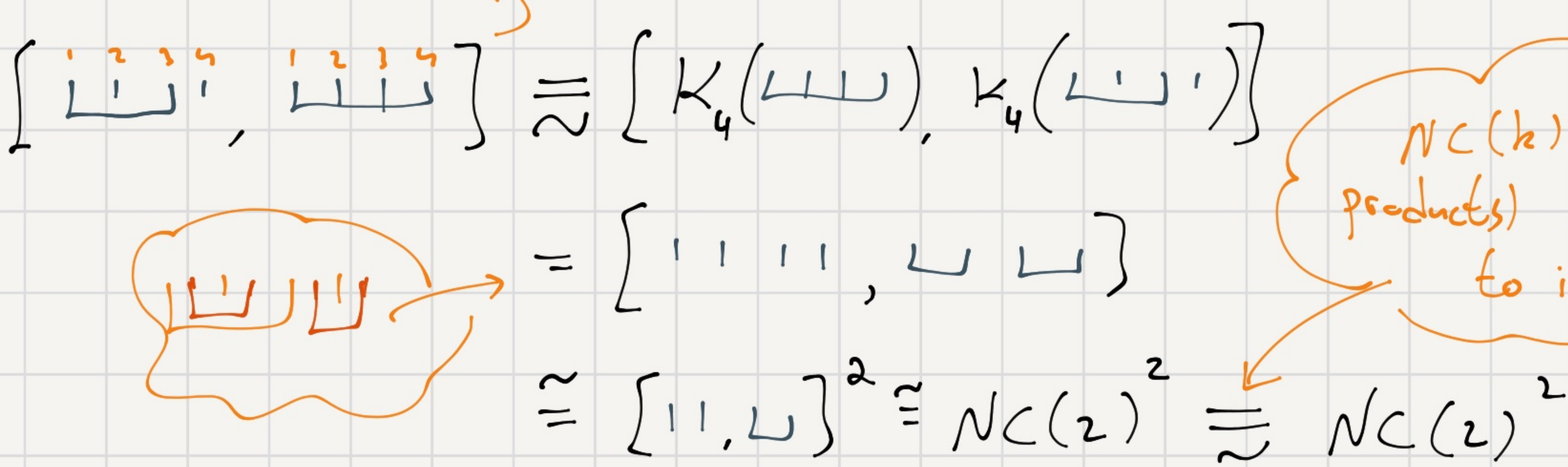
$$[\sigma, \pi] \cong P(n_1) \times \cdots \times P(n_b)$$

Does a similar decomposition hold for NC?

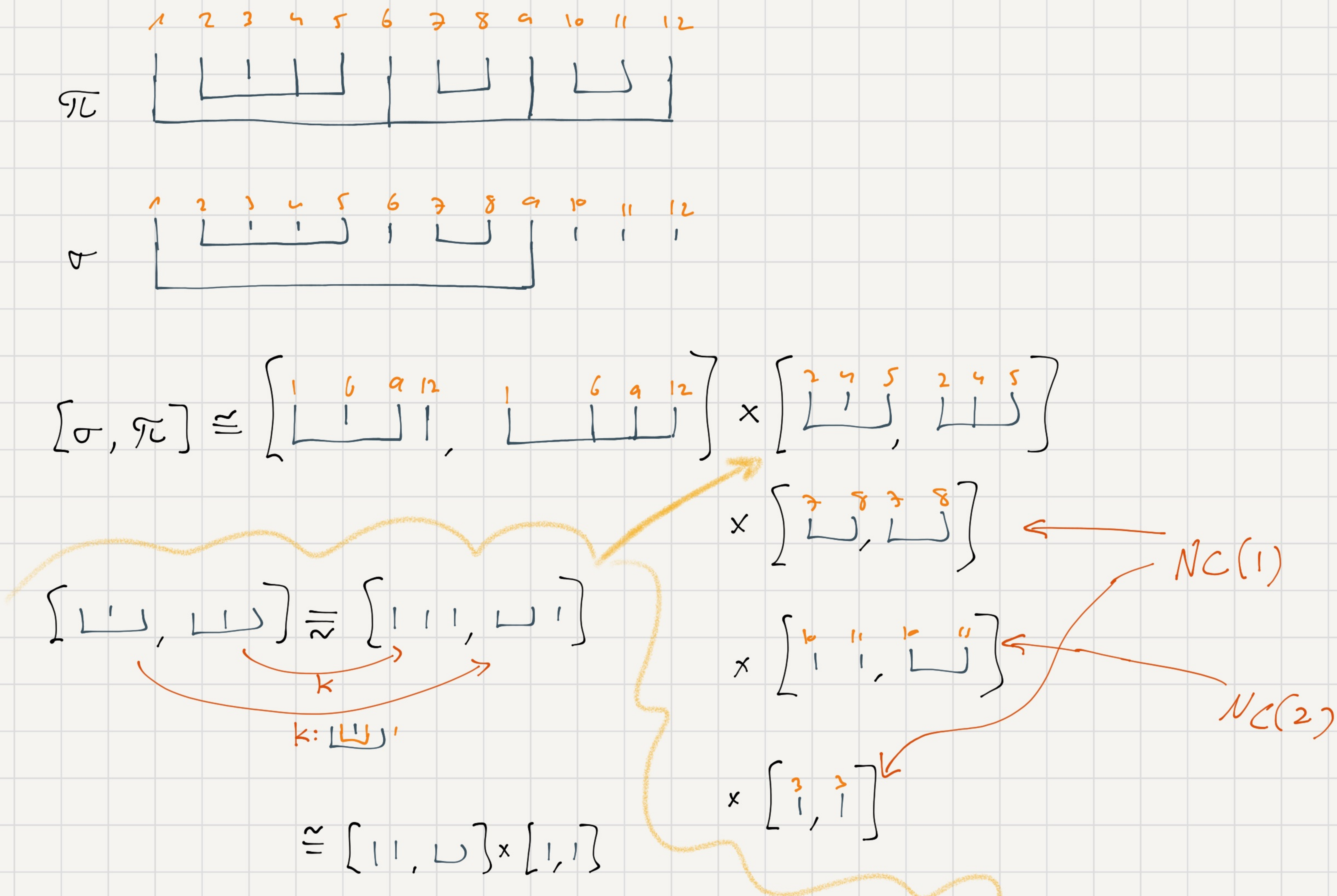


My ☺ cool notation for anti-isomorphic

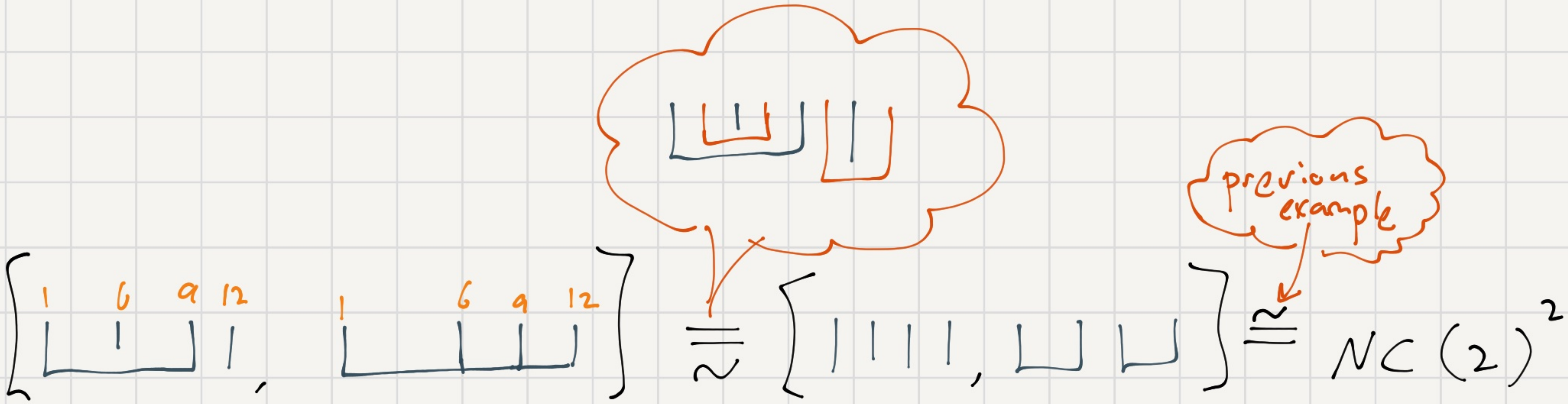




NC(k) (& their products) is anti isomorphic to itself!



$$\cong NC(2) \times NC(1) \cong NC(2) \cong NC(2).$$



$$\Rightarrow \quad [\sigma, \pi] \cong NC(2)^4$$

Theorem. $\forall\, \sigma \leq \pi$ in $NC(n)$ $\exists$ a canonical sequence $k_1, \dots, k_n$ of non-negative integers s.t.

$$[\sigma, \pi] = NC(1)^{k_1} \times NC(2)^{k_2} \times \cdots \times NC(n)^{k_n}$$

Pf.

$$[\sigma, \pi] \cong \prod_{V \in \pi} [\sigma|_V, \pi|_V]$$

By identifying, for each $V \in \pi$, V with $[|V|]$

$$\cong \prod_{V \in \pi} [\tau_V, 1_{|V|}]$$

$$\cong \prod_{V \in \pi} \underbrace{[0_{|V|}, K(\tau_V)]}_{\cong \prod_{W \in K(\tau_V)} \underbrace{[0_{|W|}, K(\tau_V)|_W]}_{\cong NC(|W|)}}$$

Thus, $\quad [\sigma,\pi] \cong \prod_{V\in\pi} \prod_{W\in K(\tau_V)} NC(|W|)$

But $\quad$ RHS $\cong$ itself $\implies [\sigma,\pi] \cong$ RHS $\blacksquare$

Remark. The term "canonical" in the theorem's statement means there is an algorithm for producing $k_1,\dots,k_n$ given σ & π. However, the canonical nature of $k_1,\dots,k_n$ is stronger: they do not depend on the algorithm used:

won't prove. part of the end-of-the-semester topics

Theorem. Suppose $r,s\geq 1$ & $m_1,\dots,m_r$, $n_1,\dots,n_s \geq 2$ s.t.

$$NC(m_1)\times\cdots\times NC(m_r) \cong NC(n_1)\times\cdots\times NC(n_s).$$

Then, $r=s$ & $n_1,\dots,n_s$ is obtained by permuting $m_1,\dots,m_r$.