Function Fields

Unit 9

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December 3, 2024

Overview

Adjoining algebraic elements

Let F/K be a field extension, and $a \in \mathrm{F}$ algebraic over K with minimal polynomial $f(T) \in \mathrm{K}[T]$. Then,

$$\mathrm{K}(a) \cong \mathrm{K}[T] \Big/ \langle f(T) \rangle.$$

Indeed, consider the ring homomorphism

$$\begin{aligned} \varphi : \mathrm{K}[T] &\to \mathrm{K}[a] = \mathrm{K}(a) \\ T &\mapsto a \end{aligned}$$

which fixes all elements of K.

Then, $\ker \varphi$ consists of all polynomials over K that vanish at a. This ideal is generated by $f(T)$. The assertion then follows by the first isomorphism theorem.

Finite extensions are algebraic extensions

Let F/K be a field extension. Recall that

$$\text{F/K is finite} \quad \Longrightarrow \quad \text{F/K is algebraic.}$$

Indeed, if $[\mathrm{F} : \mathrm{K}] = n$ then $\forall b \in \mathrm{F}$, we have that $1, b, b^2, \ldots, b^n$ are linearly dependent over K. The (nontrivial) linear relation

$$a_0 + a_1 b + \cdots + a_n b^n = 0$$

gives rise to a (nonzero) polynomial over K with b as a root.

The converse does not hold in general.

Algebraic extensions that are finitely generated are finite

Another "finiteness condition" is saying that F is finitely generated over K, namely, $F = K(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in F$.

If F/K is a finite extension then F is finitely generated over K.

So, an algebraic extension is a weaker property than finite extension, and same holds for the finitely generated property. However, both together implies finiteness.

Claim 1

If F/K is algebraic and finitely generated then it is finite.

Algebraic extensions that are finitely generated are finite

We sketch the proof of Claim 1.

Consider first a simple extension, namely, $\mathsf{F} = \mathsf{K}(b)$ for some $b \in \mathsf{F}$.

As F/K is algebraic, b is algebraic, and so if its minimal polynomial is of degree d then $1, b, b^2, \ldots, b^d$ span F over K. Thus, $[\mathsf{F} : \mathsf{K}] \leq d$. In fact, $[\mathsf{F} : \mathsf{K}] = d$ as $1, b, b^2, \ldots, b^{d-1}$ are linearly independent over K.

Consider now the case in which $\mathsf{F} = \mathsf{K}(b_1, b_2)$ is an algebraic extension.

Then, $\mathsf{K}(b_1)/\mathsf{K}$ is simple and algebraic and so it is finite. Moreover,

$$\mathsf{K}(b_1, b_2)/\mathsf{K}(b_1) \cong \mathsf{K}(b_1)(b_2)/\mathsf{K}(b_1)$$

is also algebraic and simple and so it is finite. Thus,

$$[\mathsf{K}(b_1, b_2) : \mathsf{K}] = [\mathsf{K}(b_1, b_2) : \mathsf{K}(b_1)] \cdot [\mathsf{K}(b_1) : \mathsf{K}] < \infty.$$

The general case follows by induction.

Transcendence degree

Definition 2 (Algebraic independence)

Let F/K be a field extension. A set $T \subseteq F$ is said to be algebraically independent over K if for all distinct $t_1, \ldots, t_n \in T$ and every $f \in K[T_1, \ldots, T_n] \setminus \{0\}$ it holds that

$$f(T_1, \ldots, T_n) \neq 0.$$

An algebraically independent set T is called a maximal algebraically independent set if for every $S \supsetneq T$, S is not algebraically independent.

Lemma 3

T is a maximal algebraically independent set $\iff$ F/K(T) *is algebraic.*

Transcendence degree

Definition 4

Let F/K be a field extension. A maximal algebraically independent set $T \subseteq$ F is called a transcendence basis of F over K.

You proved in the recitation that every two transcendence bases have the same cardinality, and so the following definition is sensible.

Definition 5

Let F/K be a field extension and let $T \subseteq$ F be a maximal algebraically independent set of F/K. The size of T is called the transcendence degree of F/K and is denoted by $\mathrm{tr.deg}_K F$.

Transcendence degree

Recall our running example

$$f(x,y) = y^2 - x^3 + x \in \mathsf{K}[x,y].$$

We defined the ring

$$C_f = \mathsf{K}[x,y] \Big/ \langle f(x,y) \rangle,$$

and its fraction field

$$\mathsf{K}_f = \mathrm{Frac}\, C_f \cong \mathsf{K}(x)[y] \Big/ \langle f(x,y) \rangle.$$

Exercise. What is $\mathrm{tr.deg}_{\mathsf{K}}\, \mathsf{K}_f$? Give a transcendence basis for K_f/K.

Overview

Algebraic function fields

Definition 6 (Field of constants)

Let F/K be a field extension. The field

$$\mathsf{K}' = \{\alpha \in \mathsf{F} \mid \alpha \text{ is algebraic over } \mathsf{K}\}$$

is called the field of constants of F/K.

Note that $\mathsf{K} \subseteq \mathsf{K}' \subseteq \mathsf{F}$.

Exercise. Prove that the field of constants of $\mathsf{K}(x)/\mathsf{K}$ is K.

Algebraic function fields

We turn to define the most basic object in the course.

Definition 7 (Algebraic function fields)

A field extension F/K is an algebraic function field if

1. F/K is finitely generated.
2. The field of constants K' of F/K is equal to K.
3. $F \neq K$.

Note that $\text{tr.deg}_K F < \infty$. If $\text{tr.deg}_K F = r$ we say that F is an algebraic function field in r variables over K.

Exercise. Prove that $K(x)/K$ is an algebraic function field in one variable over K.

In this course we focus on algebraic function fields in one variable. For these we have an alternative, seemingly stronger, characterization.

Algebraic function fields

Claim 8

A field extension F/K is an algebraic function field in one variable iff the following holds:

1. $\exists x \in \mathrm{F}$ *s.t.* $[\mathrm{F} : \mathrm{K}(x)] < \infty$.
2. The field of constants K' of F/K is equal to K.
3. $\mathrm{F} \neq \mathrm{K}$.

Proof.

Since a finite extension is always finitely generated, condition (1) above implies that F is finitely generated over $\mathrm{K}(x)$. Since $\mathrm{K}(x)$ is finitely generated over K, condition (1) of the definition follows.

Algebraic function fields

Proof.

For the other direction, by condition (3) of the definition, there is $x \in F \setminus K$. By (2), x is transcendental over K.

Since F/K is an algebraic function field in one variable, $\mathrm{tr.deg}_K F = 1$ and so x constitutes a transcendence basis of F/K. Lemma 3 then implies that $F/K(x)$ is algebraic.

Now, F/K is finitely generated by condition (1), and therefore so is $F/K(x)$. Claim 1 then implies that $[F : K(x)] < \infty$. □

Function fields

From this point on, we abbreviate and say a function field instead of an "algebraic function field in one variable".

Why condition (2)?

Condition (1) essentially says we are dealing with one-dimensional objects. condition (3) avoid trivialities.

If K is algebraically closed then condition (2) is vacuously true.

Consider the rational function field $\mathbb{F}_2(x)$. If it wasn't for condition (2) then $F/\mathbb{F}_2$ where

$$F = \mathbb{F}_2(x)[y] \Big/ \langle y^2 + y + 1 \rangle$$

would have been a function field. Indeed,

- It is a field as $y^2 + y + 1$ is irreducible over $\mathbb{F}_2(x)$.
- F is generated by y over $\mathbb{F}_2(x)$.
- $F \neq \mathbb{F}_2$.

However, convince yourself that

$$F = \mathbb{F}_2(x)[y] \Big/ \langle y^2 + y + 1 \rangle \cong \left(\mathbb{F}_2[y] \Big/ \langle y^2 + y + 1 \rangle \right)(x) \cong \mathbb{F}_4(x).$$

Thus, we only added new "constants" to $\mathbb{F}_2(x)$.

Overview

Function fields and curves

If F/K is a function field, by Claim 8, $\exists x \in \mathsf{F}$ s.t. $[\mathsf{F} : \mathsf{K}(x)] < \infty$. Thus, $\mathsf{F}/\mathsf{K}(x)$ is algebraic.

Take $y \in \mathsf{F} \setminus \mathsf{K}(x)$ if such exists. Let $\varphi(T) \in \mathsf{K}(x)[T]$ be its minimal polynomial over $\mathsf{K}(x)$. Then,

$$\mathsf{K}(x, y) \cong \mathsf{K}(x)[T] \big/ \langle \varphi(T) \rangle.$$

If y happens to satisfy $\mathsf{F} = \mathsf{K}(x, y)$ then we get that

$$\mathsf{F} \cong \mathsf{K}(x)[T] \big/ \langle \varphi(T) \rangle.$$

This is a converse to the way we constructed our example:

$$\mathsf{K}_f = \mathrm{Frac}\, C_f \cong \mathsf{K}(x)[y] \big/ \langle y^2 - x^3 + x \rangle.$$

As a side remark, in characteristic 0 we can always find y as above, and more generally, whenever $\mathsf{F}/\mathsf{K}(x)$ is a finite separable extension.

Function fields and curves

Observe that

$$K(x, y) = K\left(\frac{y}{x}, x\right)$$

and since $y^2 = x^3 - x$ we get

$$\left(\frac{y}{x}\right)^2 = x - \frac{1}{x}.$$

Thus, if we denote $z = \frac{y}{x}$ then $K(x, y) = K(x, z)$ and

$$z^2 = x - \frac{1}{x}.$$

Equivalently,

$$xz^2 = x^2 - 1.$$

So, two different curves may share the same function field.

Overview

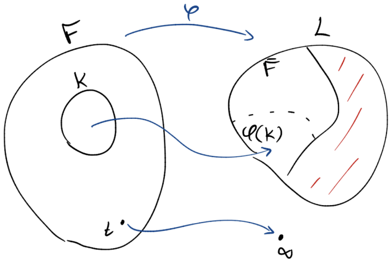
Places of function fields

Definition 9

Let F/K be a function field. A place of F/K is a nontrivial place $\varphi : F \to L \cup \{\infty\}$ that is trivial on K.

Claim 10

Let F/K be a function field and $\varphi : F \to L \cup \{\infty\}$ a place of F/K with residue field $\bar{F} = \varphi(F) \setminus \{\infty\}$. Then, $[\bar{F} : \varphi(K)] < \infty$.

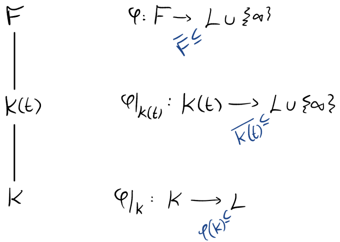


Proof.

As φ is nontrivial on F, we can pick $t \in \mathrm{F}$ with $\varphi(t) = \infty$.

Since φ is trivial on K, we have that $t \notin \mathrm{K}$. Thus, by condition (2), t is transcendental over K.

Consider the subfield $\mathrm{K}(t)$ of F which, note, is isomorphic to the field of rational functions over K. As F/K is a function field (namely, of one variable), $\{t\}$ is a transcendence basis of F over K. Hence, by Lemma 3, $\mathrm{F}/\mathrm{K}(t)$ is algebraic. Further, $\mathrm{F}/\mathrm{K}(t)$ is finitely generated (as F/K is). Thus, by Claim 1, $[\mathrm{F} : \mathrm{K}(t)] < \infty$.



Places of function fields

Proof.

In the recitations you will characterize the places of the rational function field $\mathsf{K}(t)/\mathsf{K}$ and prove that all such places have a residue field which is a finite extension of K.

The restriction $\varphi|_{\mathsf{K}(t)}$ is such a place. Thus, $[\overline{\mathsf{K}(t)} : \mathsf{K}] < \infty$. Moreover, $\mathsf{K} \cong \varphi(\mathsf{K})$, and so

$$[\bar{\mathsf{F}} : \varphi(K)] = [\bar{\mathsf{F}} : \overline{\mathsf{K}(t)}] \cdot [\overline{\mathsf{K}(t)} : \mathsf{K}].$$

In the recitations you will prove that for every field extension E/L, and a place ψ of E,

$$[\bar{\mathsf{E}} : \bar{\mathsf{L}}] \cdot (v_\psi(\mathsf{E}^\times) : v_\psi(\mathsf{L}^\times)) \leq [\mathsf{E} : \mathsf{L}].$$

Taking $\mathsf{E} = \mathsf{F}$ and $\mathsf{L} = \mathsf{K}(t)$, we conclude that

$$[\bar{\mathsf{F}} : \overline{\mathsf{K}(t)}] \leq [\mathsf{F} : \mathsf{K}(t)]$$

which recall is finite. □

Places of function fields

Definition 11

Let F/K be a function field. Let $\varphi : F \to L \cup \{\infty\}$ be a place of F/K. Then, $[\bar{F} : K]$ is called the degree of φ, and is denoted by $\deg \varphi$ (note that we identify $\varphi(K)$ with K.)

Claim 12

If φ, φ' are equivalent places of F/K then $\deg \varphi = \deg \varphi'$.

Proof.

We saw that the residue field of a valuation φ is given by

$$\bar{F}_\varphi = \mathcal{O}_\varphi \Big/ \mathfrak{m}_\varphi,$$

and we proved that for φ, φ' equivalent it holds that

$$\mathcal{O}_\varphi = \mathcal{O}_{\varphi'} \quad (\text{and so } \mathfrak{m}_\varphi = \mathfrak{m}_{\varphi'}).$$

□

Places of function fields

Definition 13

Let F/K be a function field. A valuation v on F that corresponds to a place φ of F/K is said to be a valuation of F/K.

Note that a valuation on F is a valuation of F/K iff $v(K^{\times}) = 0$ and $v(x) \neq 0$ for some $x \in$ F.

Definition 14

A place of a function field F/K is discrete if its corresponding valuation is discrete.

Theorem 15

All places of a function field are discrete.

Places of function fields

To prove Claim 15, recall a result we proved:

Claim 16

Let $\Delta \leq \Gamma$ be ordered groups. Assume that $(\Gamma : \Delta) < \infty$. Then,

$$\Delta \cong \mathbb{Z} \quad \Longrightarrow \quad \Gamma \cong \mathbb{Z}.$$

Proof.

Let φ be a place of F/K. Take $t \in \mathsf{F} \setminus \mathsf{K}$. Then, t is transcendental over K and $[F : \mathsf{K}(t)] < \infty$ as follows by the proof of Claim 10. By the theorem that you will see in the recitations,

$$(v_\varphi(\mathsf{F}^\times) : v_\varphi(\mathsf{K}(t)^\times)) \leq [F : \mathsf{K}(t)] < \infty.$$

You will further prove that all valuations of $\mathsf{K}(t)/\mathsf{K}$ are discrete. Thus, by Claim 16, φ is discrete.

Places of function fields

Here is another result you will prove in the recitations.

Claim 17

Let F/K be a function field and let $x \in F \setminus K$. Then, there are valuations v, v' of F/K with $v(x) > 0$ and $v'(x) < 0$.

This should be read as saying that every non-constant function has a zero and a pole.