Algebraic-Geometric Codes	**Fall 2024/5**
Problem Set 1	
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Problem 1. *Let k be an algebraically closed field. Find all the valuations $\nu\colon k \to \mathbb{Z} \cup \{\infty\}$.*

Problem 2. *For $i = 1, 2$, let $\mathcal{O}_i$ be a valuation ring with fraction field F and corresponding non-trivial valuation $\nu_i\colon F \to \mathbb{Z} \cup \{\infty\}$ (such a ring is called a* discrete *valuation ring).*

(a) Prove that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ if and only if for every $a \in F$,

$$\nu_2(a) > 0 \implies \nu_1(a) > 0.$$

(b) Prove that

$$\mathcal{O}_1 \subseteq \mathcal{O}_2 \implies \mathcal{O}_1 = \mathcal{O}_2.$$

Hint: *If $a, b \in F^\times$ satisfy $\nu_2(a) \geq 0$ and $\nu_2(b) > 0$, what can you say about $\nu_1(a^m b)$ and $\nu_2(a^m b)$, for $m \in \mathbb{N}^+$?*

Problem 3. *Consider the polynomial $f(x, y) = y^2 + 4x^2 + 1 \in \mathbb{F}_5[x, y]$.*

(a) Show that $\langle f \rangle \in \operatorname{Spec}(\mathbb{F}_5[x, y])$.

(b) Let $C_f := {}^{\mathbb{F}_5[x,y]}\!/\!_{\langle f \rangle}$. Then C_f is an integral domain. Denote its fraction field by $K_f := \operatorname{Frac}(C_f)$. Explain why every element $g \in K_f$ can be written as

$$g = A(x) + B(x)y \tag{1}$$

where $A(t), B(t) \in \mathbb{F}_5(t)$ (with the usual abuse of notation of x, y denoting the cosets of x and y in C_f).

(c) The point $\mathfrak{p} = (1, 0) \in \mathbb{F}_5^2$ is on the curve Z_f (as $f(1, 0) = 0$). Provided that there is a corresponding valuation $\nu_\mathfrak{p}\colon K_f \to \mathbb{Z} \cup \{\infty\}$ with $\nu_\mathfrak{p}(x - 1) = 2$, compute the values

- $\nu_\mathfrak{p}(y)$
- $\nu_\mathfrak{p}(p(x))$ *for a non-zero polynomial $p(x) = \sum_{i=0}^{d} a_i (x-1)^i \in \mathbb{F}_5[x]$*

(d) Conclude that if $g \in K_f$ admits the representation (1), then

$$\nu_\mathfrak{p}(g) = \min\left(\nu_\mathfrak{p}(A(x)), 1 + \nu_\mathfrak{p}(B(x))\right).$$

(e) What is the corresponding valuation ring $\mathcal{O}_\mathfrak{p}$? Justify your answer.

(f) What is its maximal ideal $\mathfrak{m}_\mathfrak{p}$ and a corresponding place $\psi_\mathfrak{p}$? no justification is needed.

Problem 4. *Let L/K be field extension, $S \subseteq L$ algebraically independent over K and $a \in L$. Prove that*

$$a \text{ is algebraic over } K(S) \implies S \cup \{a\} \text{ is algebraically dependent over } K$$

(in the recitation we will prove the converse).