The co-norm

Unit 20

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The co-norm

Throughout this unit, we let F/L be a function field extension of E/K.

Definition 1

Let $\mathfrak{p}$ be a prime divisor of E/K. To $\mathfrak{p}$ we associate a divisor

$$\mathrm{Con}(\mathfrak{p}) = \sum_{\mathfrak{P}/\mathfrak{p}} e(\mathfrak{P}/\mathfrak{p})\mathfrak{P}.$$

The map Con can be extended in the natural way to a group homomorphism

$$\mathrm{Con} : \mathcal{D}(\mathrm{E}/\mathrm{K}) \to \mathcal{D}(\mathrm{F}/\mathrm{L})$$

We call Con the co-norm and sometimes denote it by $\mathrm{Con}_{\mathrm{F}/\mathrm{E}}$.

The co-norm

The following are straightforward properties of the co-norm.

Lemma 2

1. $\mathfrak{a} \geq 0 \implies \mathrm{Con}(\mathfrak{a}) \geq 0$.
2. *If* $\mathfrak{a}, \mathfrak{b}$ *are disjoint then so are* $\mathrm{Con}(\mathfrak{a}), \mathrm{Con}(\mathfrak{b})$.
3. *If* F'/L' *is an extension of* F/L *then*
$$\mathrm{Con}_{F'/E} = \mathrm{Con}_{F'/F} \circ \mathrm{Con}_{F/E}.$$
4. Con *is one to one.*

The co-norm

The co-norm is a natural homomorphism in that it “lifts” a principle divisor of E to the corresponding divisor in F.

Lemma 3

For every $x \in \mathsf{E}^{\times}$,

$$(x)_{\mathsf{F}} = \mathrm{Con}((x)_{\mathsf{E}}),$$
$$(x)_{\mathsf{F},0} = \mathrm{Con}((x)_{\mathsf{E},0}),$$
$$(x)_{\mathsf{F},\infty} = \mathrm{Con}((x)_{\mathsf{E},\infty}).$$

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Proof.

We start by proving that $(x)_{\mathsf{F}} = \mathrm{Con}((x)_{\mathsf{E}})$.

$$\begin{aligned}
(x)_{\mathsf{F}} &= \sum_{\mathfrak{P} \in \mathbb{P}(\mathsf{F}/\mathsf{L})} \upsilon_{\mathfrak{P}}(x)\mathfrak{P} \\
&= \sum_{\mathfrak{p} \in \mathbb{P}(\mathsf{E}/\mathsf{K})} \sum_{\mathfrak{P}/\mathfrak{p}} \upsilon_{\mathfrak{P}}(x)\mathfrak{P} \\
&= \sum_{\mathfrak{p} \in \mathbb{P}(\mathsf{E}/\mathsf{K})} \upsilon_{\mathfrak{p}}(x) \sum_{\mathfrak{P}/\mathfrak{p}} e(\mathfrak{P}/\mathfrak{p})\mathfrak{P} \\
&= \sum_{\mathfrak{p} \in \mathbb{P}(\mathsf{E}/\mathsf{K})} \upsilon_{\mathfrak{p}}(x)\mathrm{Con}(\mathfrak{p}) \\
&= \mathrm{Con}\left(\sum_{\mathfrak{p} \in \mathbb{P}(\mathsf{E}/\mathsf{K})} \upsilon_{\mathfrak{p}}(x)\mathfrak{p} \right) \\
&= \mathrm{Con}((x)_{\mathsf{E}}).
\end{aligned}$$

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Proof.

This proves the first item. Using it, we have that

$$(x)_{\mathsf{F}} = \mathrm{Con}((x)_{\mathsf{E}}) = \mathrm{Con}((x)_{\mathsf{E},0}) - \mathrm{Con}((x)_{\mathsf{E},\infty}).$$

By the Lemma 2, the two divisors $\mathrm{Con}((x)_{\mathsf{E},0})$ and $\mathrm{Con}((x)_{\mathsf{E},\infty})$ are non-negative and disjoint, and the proof of items 2,3 follow. $\square$

Some notation

When we take a divisor $\mathfrak{a} \in \mathcal{D}(\mathsf{E}/\mathsf{K})$ and would like to consider "it" over F/L we will identify $\mathfrak{a}$ with $\mathrm{Con}(\mathfrak{a})$. To keep notation light, we will sometimes use $\mathfrak{a}$ rather than $\mathrm{Con}(\mathfrak{a})$.

However, the degree and dimension of $\mathfrak{a}$ and $\mathrm{Con}(a)$ typically will not be the same, and so we write $\deg_{\mathsf{F}} \mathfrak{a}$ for $\deg \mathrm{Con}(\mathfrak{a})$ and $\deg_{\mathsf{E}}(\mathfrak{a})$ for $\deg \mathfrak{a}$, and similarly define $\dim_{\mathsf{F}} \mathfrak{a}$ as $\dim \mathrm{Con}(\mathfrak{a})$.

Degree under co-norm

Theorem 4

There exists a constant λ *that depends only on* F/E *s.t.* $\forall \mathfrak{a} \in \mathcal{D}(\mathsf{E}/\mathsf{K})$,

$$\deg_{\mathsf{F}} \mathfrak{a} = \lambda \cdot \deg_{\mathsf{E}} \mathfrak{a}.$$

Moreover, if F/E *is finite then* $\lambda = \frac{[\mathsf{F}:\mathsf{E}]}{[\mathsf{L}:\mathsf{K}]}$.

Proof.

We first prove the theorem for the case that F/E is finite. It suffices to prove the theorem for prime divisors (as $\deg_{\mathsf{E}}$ and $\deg_{\mathsf{F}} = \deg \circ \mathrm{Con}$ are homomorphisms).

Degree under co-norm

Proof.

Take $\mathfrak{p} \in \mathcal{D}(E/K)$. Then,

$$\begin{aligned}\deg_F \mathfrak{p} &= \deg \mathrm{Con}(\mathfrak{p}) \\ &= \deg \sum_{\mathfrak{P}/\mathfrak{p}} e(\mathfrak{P}/\mathfrak{p})\mathfrak{P} \\ &= \sum_{\mathfrak{P}/\mathfrak{p}} e(\mathfrak{P}/\mathfrak{p}) \deg \mathfrak{P}.\end{aligned}$$

Recall that

$$[L:K] \deg \mathfrak{P} = f(\mathfrak{P}/\mathfrak{p}) \deg_E \mathfrak{p},$$

and so, using the fundamental equality,

$$\deg_F \mathfrak{p} = \frac{\deg_E \mathfrak{p}}{[L:K]} \sum_{\mathfrak{P}/\mathfrak{p}} e(\mathfrak{P}/\mathfrak{p}) f(\mathfrak{P}/\mathfrak{p}) = \frac{[F:E]}{[L:K]} \cdot \deg_E \mathfrak{p}.$$

Degree under co-norm

Proof.

We turn to consider the general case. It suffices to show that for all prime divisors $\mathfrak{p}, \mathfrak{q}$,

$$\frac{\deg_{\mathsf{E}} \mathfrak{p}}{\deg_{\mathsf{F}} \mathfrak{p}} = \frac{\deg_{\mathsf{E}} \mathfrak{q}}{\deg_{\mathsf{F}} \mathfrak{q}}.$$

Assume towards a contradiction that for some prime divisors $\mathfrak{p}, \mathfrak{q}$ it holds that

$$\frac{\deg_{\mathsf{E}} \mathfrak{p}}{\deg_{\mathsf{F}} \mathfrak{p}} < \frac{\deg_{\mathsf{E}} \mathfrak{q}}{\deg_{\mathsf{F}} \mathfrak{q}},$$

or, equivalently,

$$\frac{\deg_{\mathsf{E}} \mathfrak{p}}{\deg_{\mathsf{E}} \mathfrak{q}} < \frac{\deg_{\mathsf{F}} \mathfrak{p}}{\deg_{\mathsf{F}} \mathfrak{q}}.$$

Degree under co-norm

Proof.

$$\frac{\deg_{\mathsf{E}} \mathfrak{p}}{\deg_{\mathsf{E}} \mathfrak{q}} < \frac{\deg_{\mathsf{F}} \mathfrak{p}}{\deg_{\mathsf{F}} \mathfrak{q}}.$$

Then, there are $m, n \in \mathbb{N}$ s.t.

$$\frac{\deg_{\mathsf{E}} \mathfrak{p}}{\deg_{\mathsf{E}} \mathfrak{q}} < \frac{m}{n} < \frac{\deg_{\mathsf{F}} \mathfrak{p}}{\deg_{\mathsf{F}} \mathfrak{q}}$$

Equivalently,

$$\deg_{\mathsf{F}}(n\mathfrak{p} - m\mathfrak{q}) > 0,$$
$$\deg_{\mathsf{E}}(n\mathfrak{p} - m\mathfrak{q}) < 0.$$

Degree under co-norm

Proof.

$$\deg_{\mathsf{F}}(n\mathfrak{p} - m\mathfrak{q}) > 0,$$
$$\deg_{\mathsf{E}}(n\mathfrak{p} - m\mathfrak{q}) < 0.$$

To get a contradiction, it suffices to prove that

$$\forall \mathfrak{a} \in \mathcal{D}(\mathsf{E}/\mathsf{K}) \qquad \deg_{\mathsf{E}} \mathfrak{a} > 0 \quad \Longrightarrow \quad \deg_{\mathsf{F}} \mathfrak{a} \geq 0.$$

Consider $k \in \mathbb{N}$ sufficiently large (compared to g_{E}). By Riemann-Roch,

$$\dim_{\mathsf{E}} k\mathfrak{a} = k \deg_{\mathsf{E}} \mathfrak{a} + 1 - g_{\mathsf{E}} > 0.$$

Thus, $\exists x \in \mathsf{E}^{\times}$ s.t. $(x)_{\mathsf{E}} + k\mathfrak{a} \geq 0$. By Lemma 2 and Lemma 3,

$$0 \leq \mathrm{Con}((x)_{\mathsf{E}} + k\mathfrak{a}) = (x)_{\mathsf{F}} + k\mathrm{Con}(\mathfrak{a}),$$

and so, as $\deg((x)_{\mathsf{F}}) = 0$,

$$\deg_{\mathsf{F}} \mathfrak{a} = \deg \mathrm{Con}(\mathfrak{a}) \geq 0.$$