Complex Analysis

101

Holomorphic Functions

recall

Def. $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 if it is approximately linear:

$$f(x) = f(x_0) + a(x - x_0) + \text{small error}$$

$$\frac{\text{error}}{x - x_0} \xrightarrow[x \to x_0]{} 0$$

Alternatively (and equivalently)

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists}$$

$w = u + iv \quad z = x + iy$ We think of w & z as bivariate real functions, ignoring the $\mathbb{C}$-structure.

$$\underbrace{\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}}_{w} = \underbrace{\begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}_{\text{linear function}} + \text{error}$$

$$\frac{|\text{error}|}{\text{dist}((x,y),(x_0,y_0))} \xrightarrow{(x,y) \to (x_0,y_0)} 0$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Nothing complex so far, only as $\mathbb{R}^2 \to \mathbb{R}^2$.

Complex derivative

$$w(z) = w(z_0) + A(z - z_0) + \text{error}$$

If $A = a + bi$ then

$$A(z-z_0) = (a+bi)\big((x-x_0) + (y-y_0)i\big) = a(x-x_0) - b(y-y_0) + i\big(a(y-y_0) + b(x-x_0)\big)$$

In matrix form (wrt the basis $\{1, i\}$):

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} \quad \text{compare to} \quad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Hence,

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}\right\} \text{The Cauchy-Riemann equations}$$

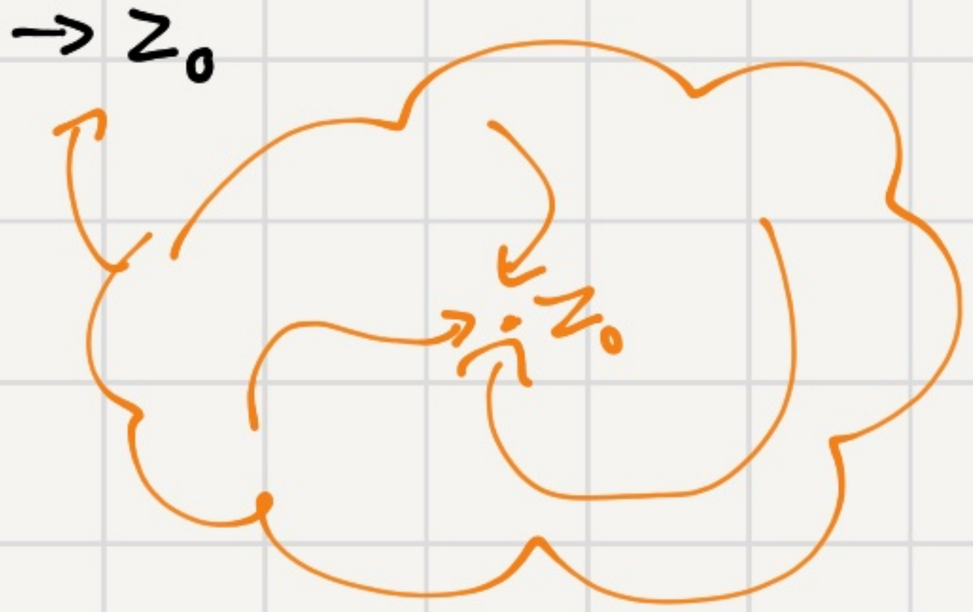
← These are the suff & nec conditions for complex diff

In terms of limits

$$A = \lim_{z \to z_0} \frac{w(z) - w(z_0)}{z - z_0}$$

doesn't make sense for $\mathbb{R}^2$

so actually encodes the $\mathbb{C}$ structure



E.g. $|z|$ is not differentiable at 0 since

$$\lim_{r \to 0^+} \frac{|ir| - 0}{ir - 0} = \frac{r}{ir} = -i \qquad \text{vs.} \qquad \lim_{r \to 0^+} \frac{|r| - 0}{r - 0} = 1$$

(check that the Cauchy-Riemann equations don't hold).

Def. A set $U \subseteq \mathbb{C}$ is open if

$$\forall u \in U \quad \exists r > 0 \quad \text{s.t.} \quad B_r(u) \subseteq U$$

$\{z \in \mathbb{C} : |z-u| < r\}$

Def. Suppose $w(z)$ is a complex function in an open set $U \subseteq \mathbb{C}$. $w(z)$ is called holomorphic if it has complex derivatives everywhere on U.

entire/whole shape ?

Def. $w(z)$ is analytic in an open set U if it has a power series expansion at each point of U.

Thm For complex functions

Once-differentiable always differentiable ?!

$$\text{Holomorphic} \iff \text{Analytic}$$

Examples $1, z$. If f, g hol $\Rightarrow$ also are $f \pm g$, fg, $\frac{f}{g}$ $g \neq 0$, $f(g(z))$

Non examples $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$, $\bar{z}$

Analytic continuation

Example of analytic continuation

Consider the power series $g(z) = \sum_{n=0}^{\infty} z^n$ which convergence in $D = \{ z \mid |z| < 1 \}$. It is given as a power series around 0, hence analytic $\Leftrightarrow$ holomorphic on D.

Can we evaluate g @ 1? 2? no! but observe that $g(z)$ agrees with $\frac{1}{1-z}$ on D. So perhaps we should define $g(2) = -1$ ($g(1)$ is still a problem).

This only makes sense if $\frac{1}{1-z}$ is in some sense the unique way of extending $g(z)$.

As it turns out, $\frac{1}{1-z}$ is indeed the unique way of extending $g(z)$ beyond D & in particular to $\mathbb{C}\setminus\{0\}$ if we wish to have an analytic ($\Leftrightarrow$ holomorphic) extension.

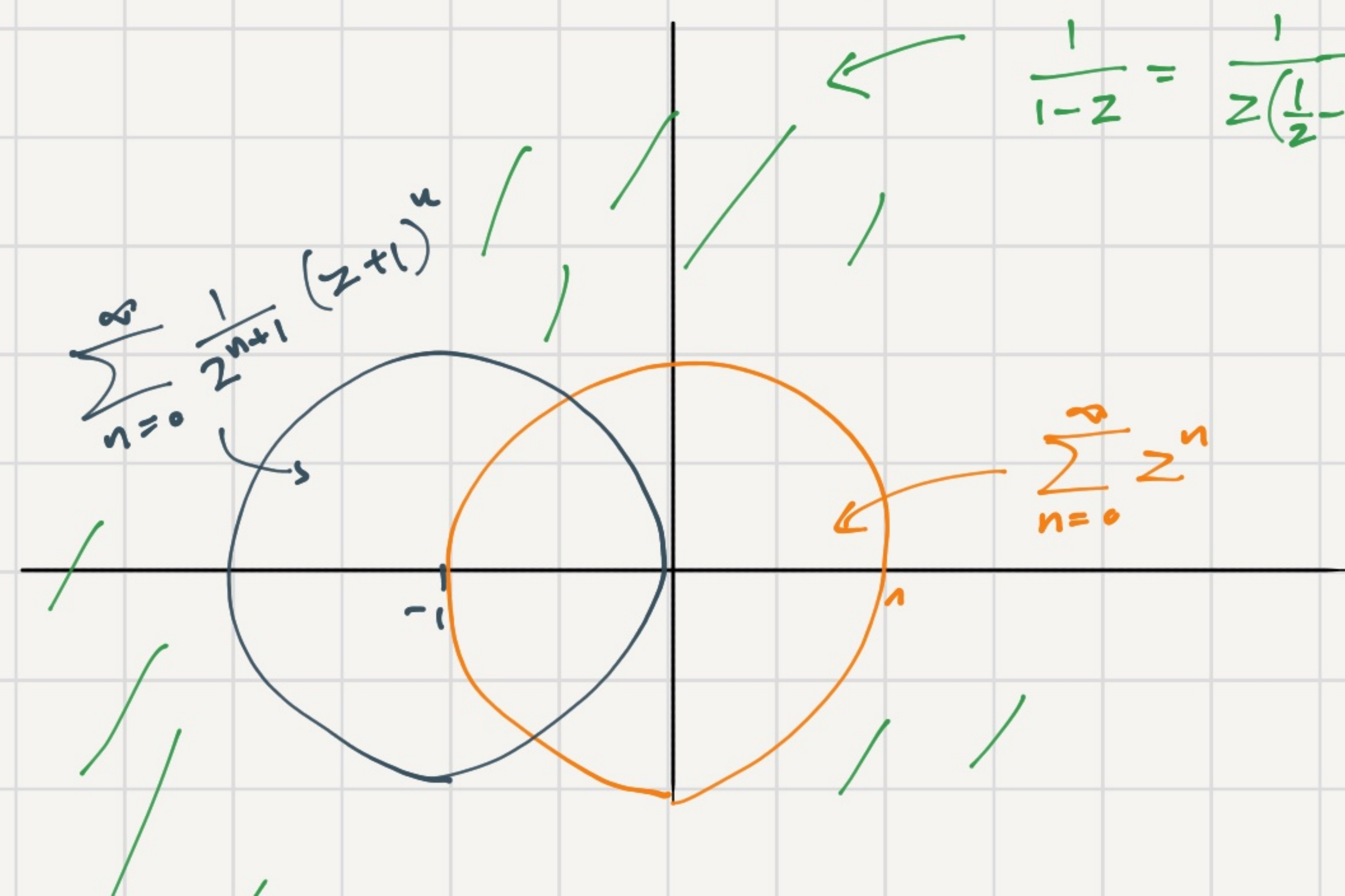
Put differently, $g(z)$ contains the same information as its global counterpart $\frac{1}{1-z}$ when insisting on analyticity.

There are of course other "local views":

For $|z|>1$

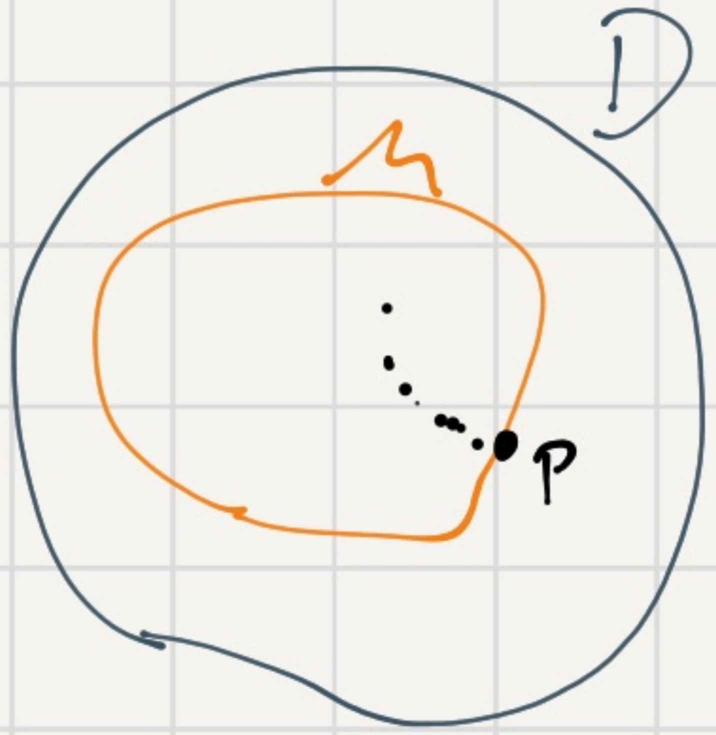
$$\frac{1}{1-z} = \frac{1}{z\left(\frac{1}{z}-1\right)} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$



not quite a Taylor expansion but is a valid series expression which we think of as Taylor expansion @ ∞

Def. Let $M \subseteq D$. A point $p \in D$ is an accumulation point of the set M if $\underbrace{\forall \varepsilon > 0 \ \exists q \in M \text{ s.t. } |p-q| < \varepsilon.}_{\iff \forall r>0 \ B_r(p) \cap M \neq \{p\}}$



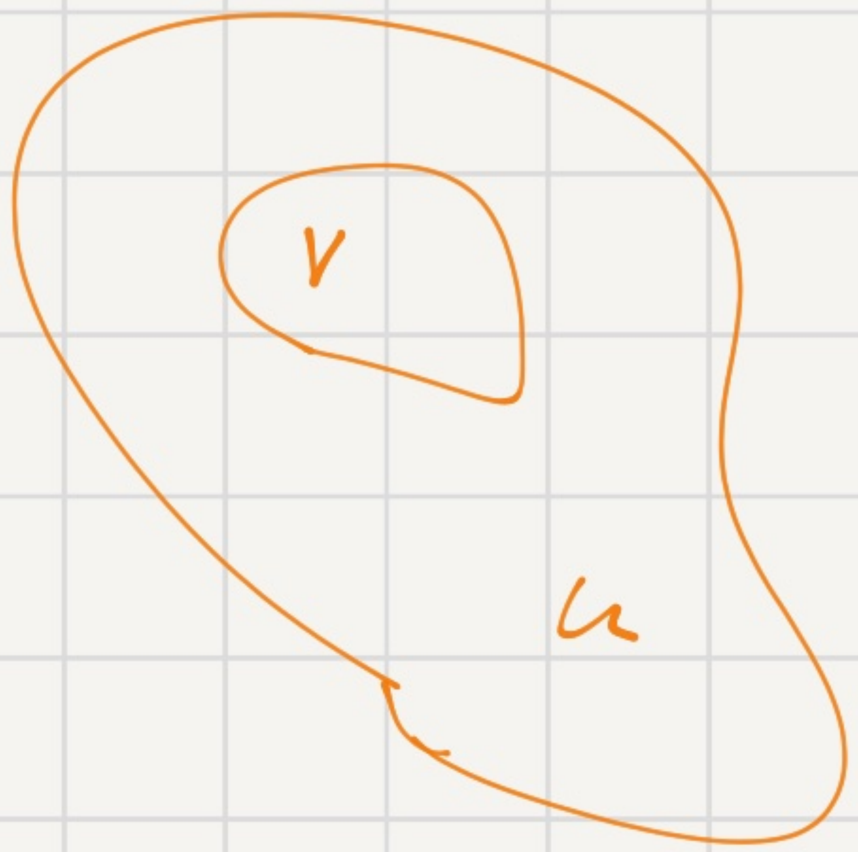
The identity theorem Let $f,g : D \to \mathbb{C}$ holomorphic. D is open & connected. ("domain") Assume $E = \{ z \in D \mid f(z) = g(z) \}$ has an accumulation point in D. Then, $f = g$ (!)

Corollary If $f,g : D \to \mathbb{C}$ are holomorphic & $f \equiv g$ on some open set $U \subseteq D$ then $f \equiv g$ on all of D.

Analytic continuation

Suppose f is holomorphic on an open set V.

Let $U \supseteq V$ open & connected. Then, $\exists$ at most one way to extend f to a holomorphic function on U. The unique function, if exists is called the analytic continuation of f to U.



Due to the identity theorem

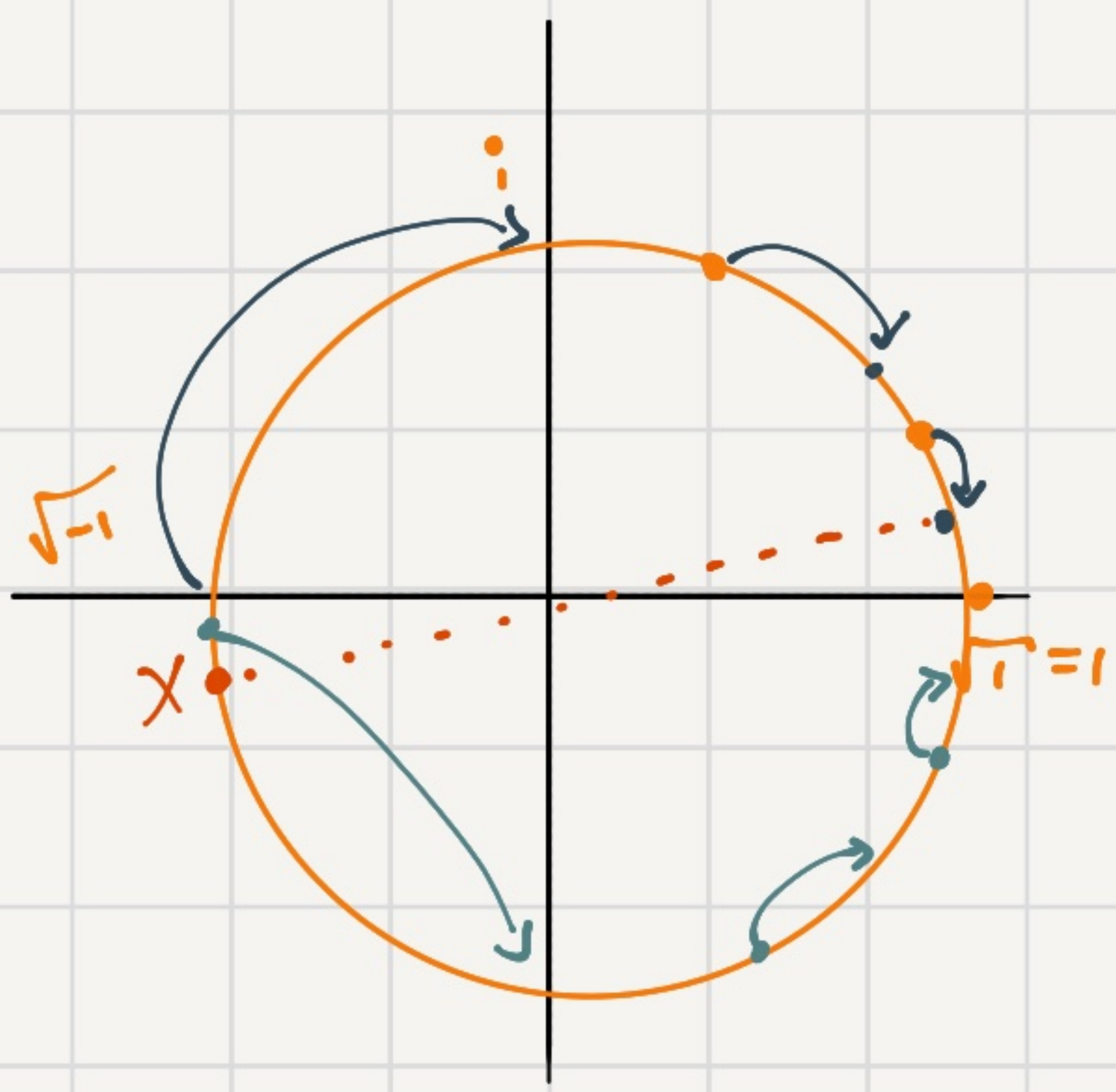
$$\sqrt{\quad}$$

&

Branch cuts

In $\mathbb{R}$ we have the convention that $\sqrt{z}$, $z>0$ is the <u>positive</u> number x s.t. $x^2 = z$. This gives a continuous function $\sqrt{\cdot} : [0,\infty) \to [0,\infty)$. In $\mathbb{C}$ things are more complicated.

<u>Problem.</u> Choose a canonical $\sqrt{z}$ to get a continuous function.

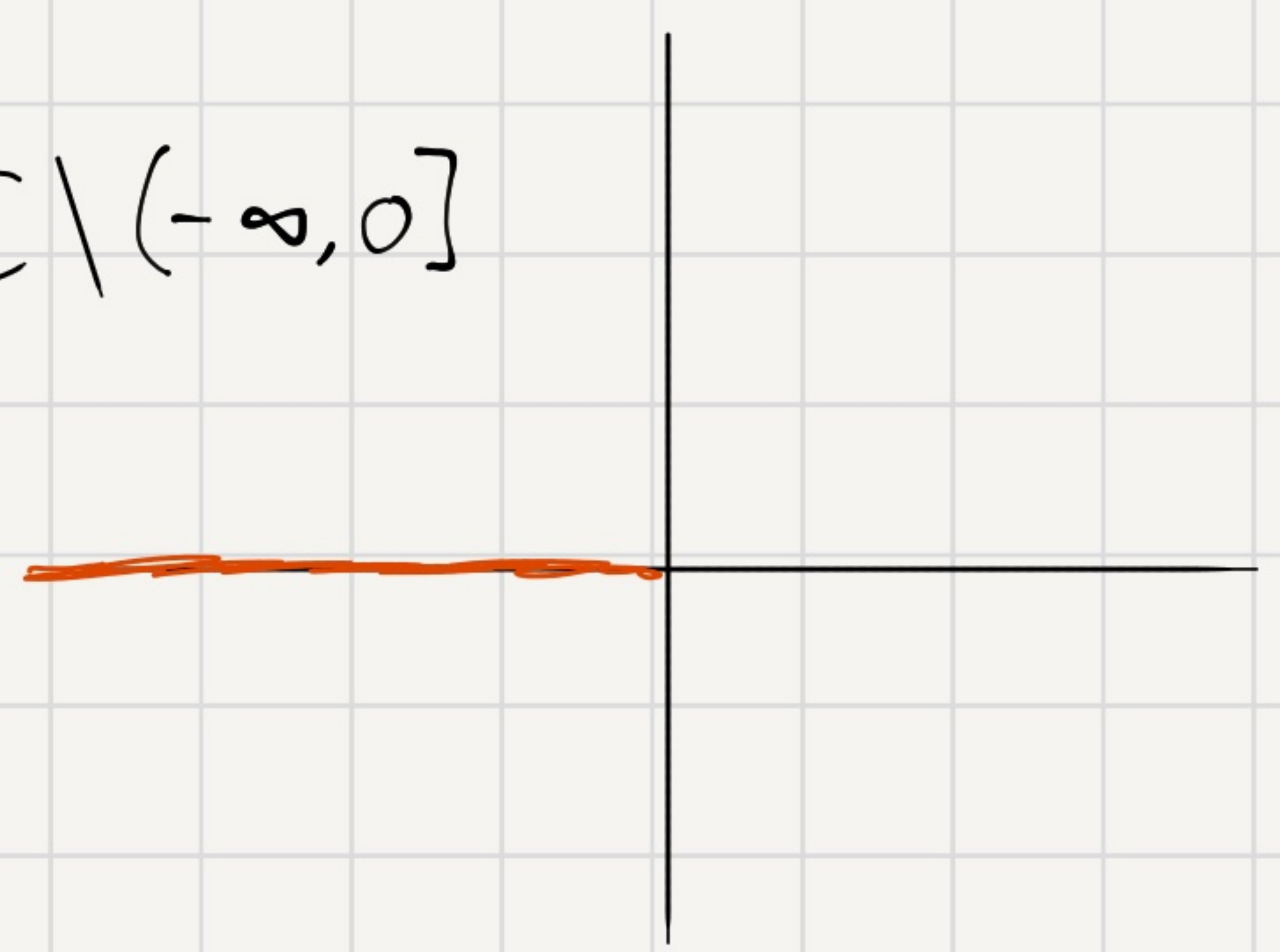


$\sqrt{-1} = i / -i$

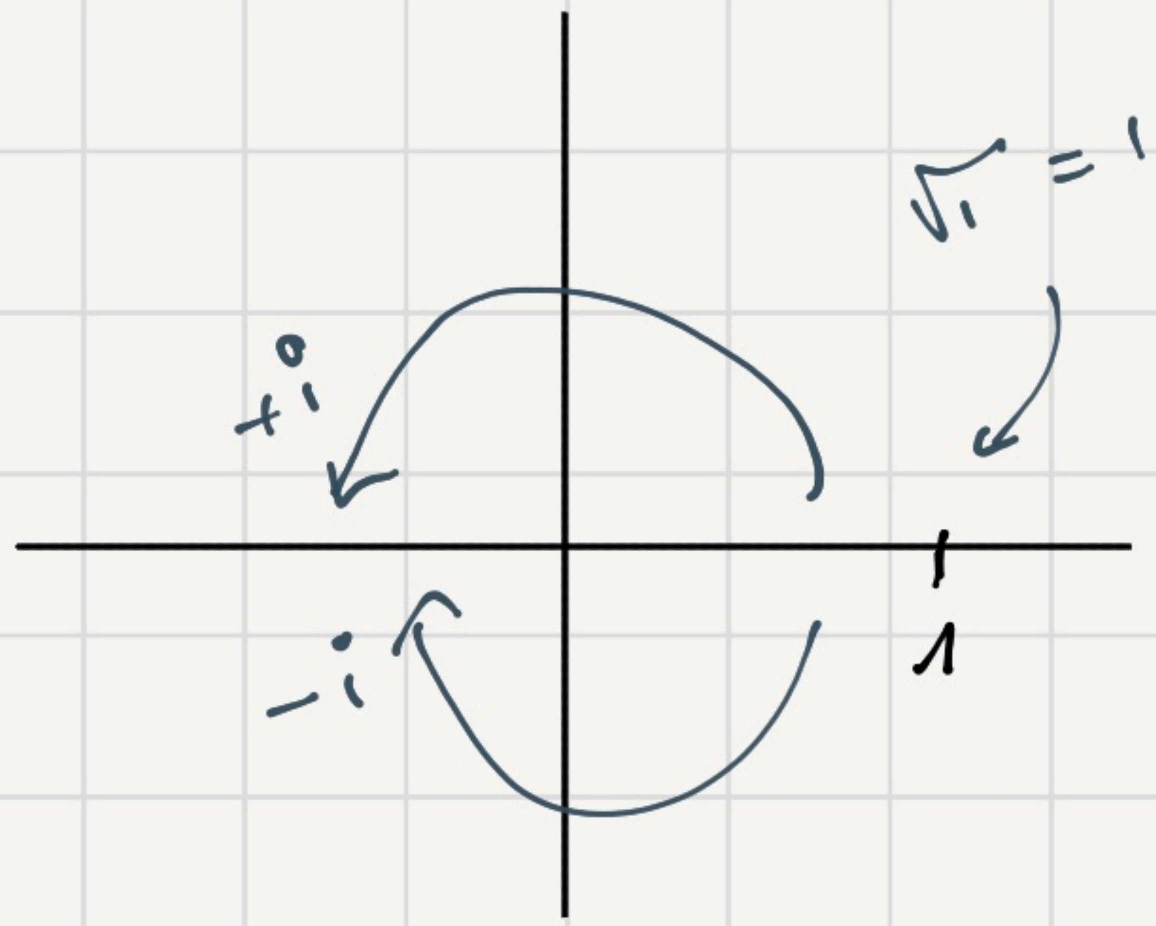
Can't make continuous!

Most common solution: branch cut

$\mathbb{C} \setminus (-\infty, 0]$

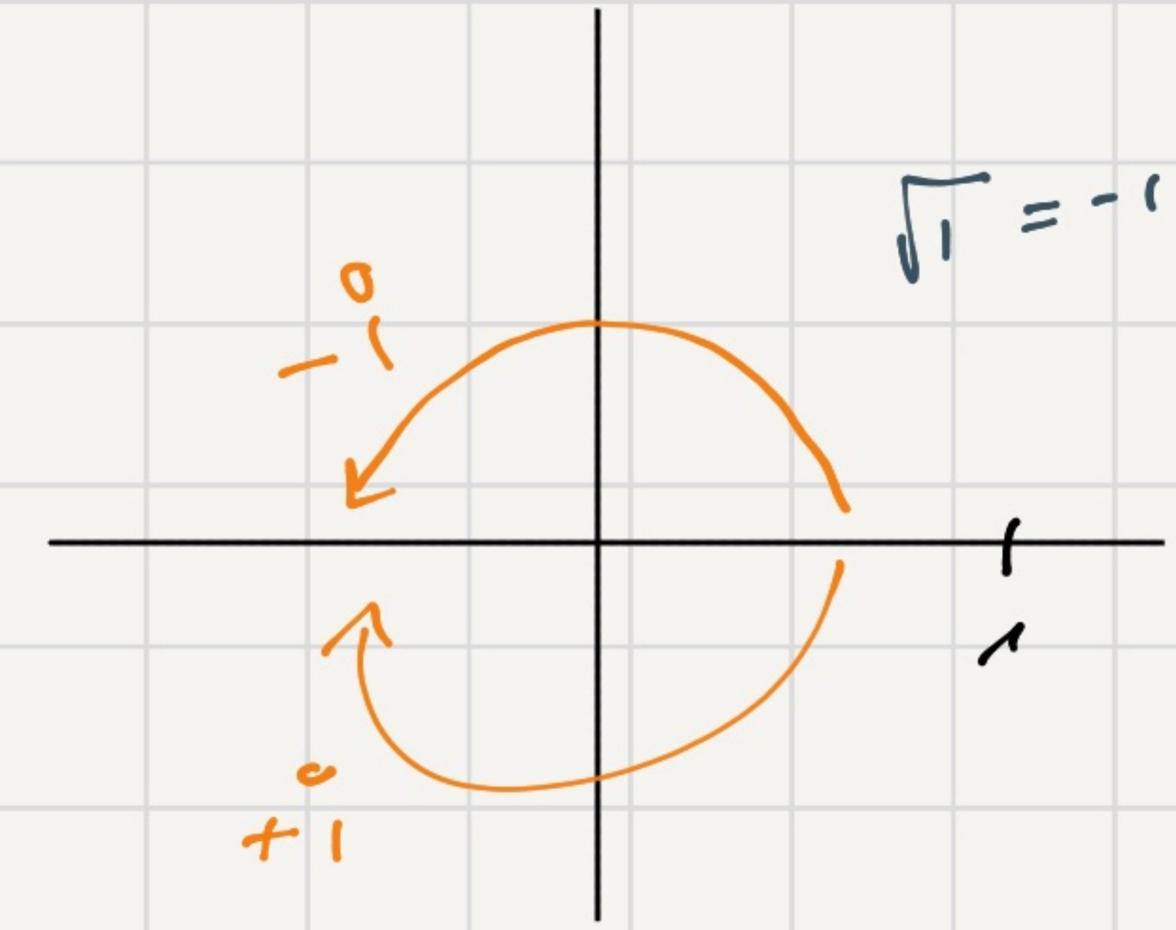


$$\sqrt{z} = \sqrt{r}\, e^{\frac{i\theta}{2}}$$

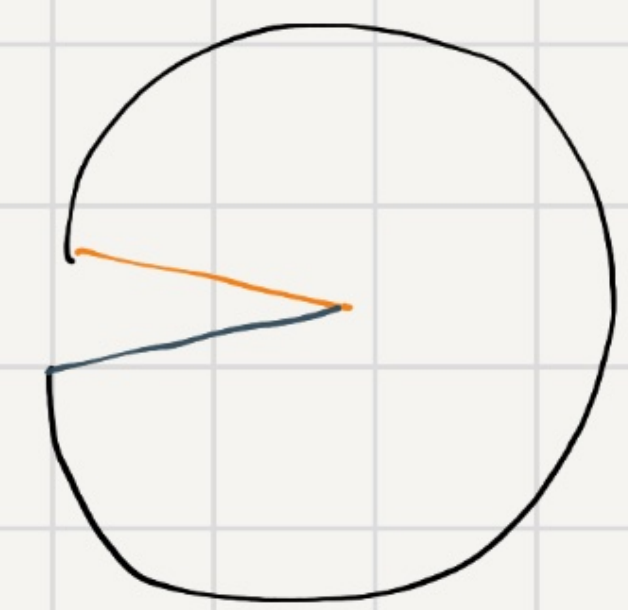


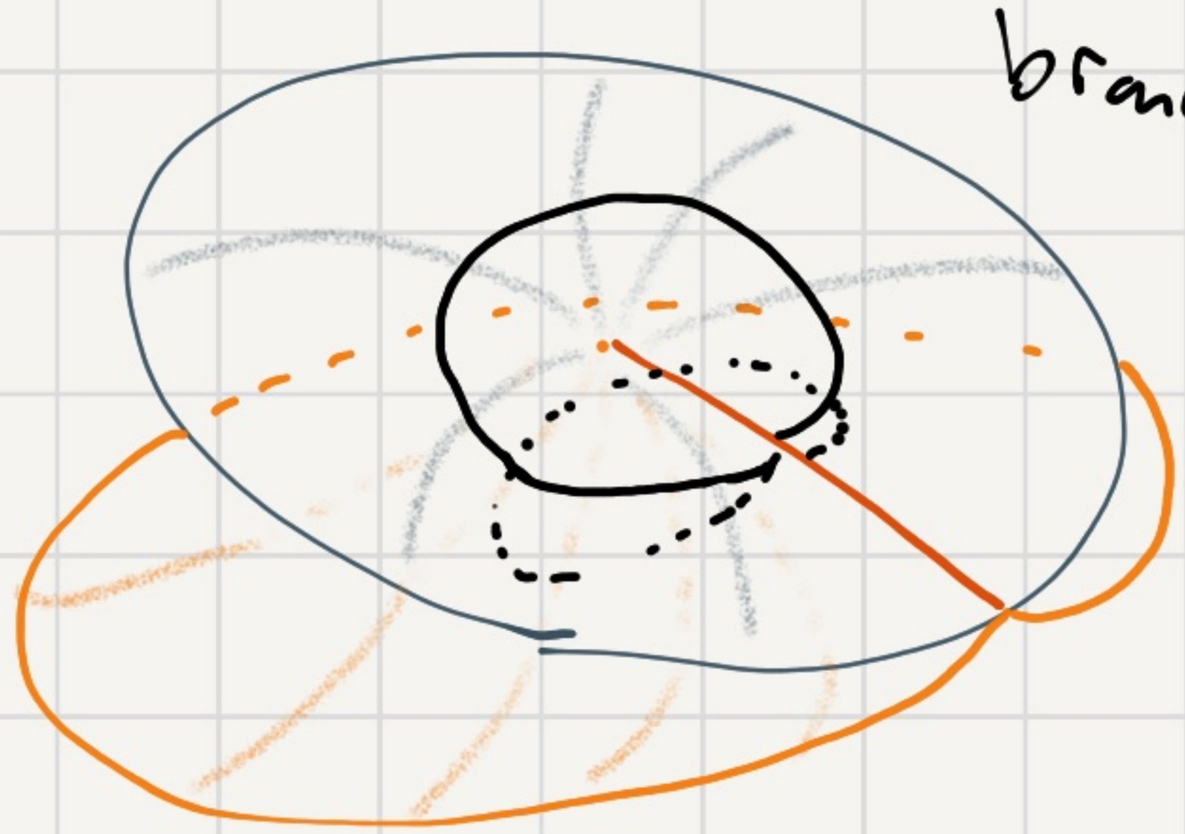
A bonus slide

$$\sqrt{z} = \sqrt{r}\, e^{i\left(\frac{\theta}{2}+\pi\right)}$$



Glue these to make $\sqrt{z}$ continuous at the branch cut





A Riemann surface