Algebraic-Geometric Codes **Fall 2024/5**

Problem Set 2

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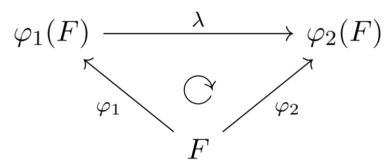
Problem 1. *Let φ_1, φ_2 be places of a field F. Show that φ_1 and φ_2 are equivalent if and only if there is a bijection $\lambda\colon \varphi_1(F) \to \varphi_2(F)$ such that $\varphi_2 = \lambda \circ \varphi_1$, $\lambda(\infty) = \infty$ and the restriction $\lambda\colon \varphi_1(F)\backslash\{\infty\} \to \varphi_2(F)\backslash\{\infty\}$ is a field isomorphism.*



Problem 2. *Let $\mathfrak{p}$ be a prime divisor and $\mathcal{O}_\mathfrak{p}$ be a valuation ring with local parameter t (i.e. $\nu_\mathfrak{p}(t) = 1$). Prove that if $0 \neq J \trianglelefteq \mathcal{O}_\mathfrak{p}$ is an ideal, then $J = t^k\mathcal{O}_\mathfrak{p}$ for some $k \in \mathbb{N}$. In particular, $\mathcal{O}_\mathfrak{p}$ is a PID.*

Problem 3. *Show that the rational function field $K(x)/K$ is indeed a function field.*

Problem 4. *Let $K = \mathbb{F}_3$ and $K(x)$ the rational function field over K.*

(a) Show that the polynomial $f(T) = T^2 + x^4 - x^2 + 1$ is irreducible over $K(x)$.

(b) Consider the quadratic extension of $K(x)$ given by $F = {}^{K(x)[y]}\big/_{\langle f(y)\rangle}$. Show that the field of constants K' of F/K has 9 elements, and $F = K'(x)$.

Hint 1. *Search for a root of $p(T) = T^2 + 1 \in K[T]$ in F.*

Hint 2. *You may use the fact that $[K(z) : K] = [K(x)(z) : K(x)]$ for every $z \in K'$.*

Hint 3. *If $z \in F$ satisfies $z^2 + bz + c = 0$ for $b, c \in K$, then by completing the square we get $\left(z + \frac{b}{2}\right)^2 = \frac{b^2}{4} - c$ (note that $K = \mathbb{F}_3$ so $\operatorname{char} K \neq 2$).*

Problem 5. *Let F/K be a function field. Prove that if $f, g \in F^\times$ satisfy $(f) = (g)$ then $f = cg$ for some $c \in K^\times$ (namely, if $f, g \in F^\times$ have the same zeros and poles, including multiplicities, then they differ by a constant factor).*

Problem 6. *Let $\mathfrak{a}$ be a divisor.*

(a) Assume $\deg \mathfrak{a} = 0$. Prove that the following are equivalent:

(i). $\mathfrak{a}$ *is principal.*

(ii). $\dim \mathfrak{a} \geq 1$.

(iii). $\dim \mathfrak{a} = 1$.

(b) Conclude that if $\deg \mathfrak{a} = 0$ and $\mathfrak{a}$ is not principal, then $\dim \mathfrak{a} = 0$, and that if $\mathfrak{a}$ is principal then $\dim \mathfrak{a} = 1$ and $\deg \mathfrak{a} = 0$.

Problem 7. *Let* $\mathfrak{a} \in \mathcal{D}$.

(a) Show that if $\deg \mathfrak{a} \geq 0$ *then* $\dim \mathfrak{a} \leq \deg \mathfrak{a} + 1$.

(b) Prove that if $\mathfrak{a} \geq 0$ *and* $k \in \mathbb{N}^+$, *then*

$$\dim((k-1)\mathfrak{a}) \leq \dim(k\mathfrak{a}) \leq \dim((k-1)\mathfrak{a}) + \deg \mathfrak{a}.$$

Problem 8. *Let* K *be an* infinite *field and let* F/K *be a function field.*

(a) Let $\mathfrak{a}, \mathfrak{b} \geq 0$. *Show that* $\dim \mathfrak{a} + \dim \mathfrak{b} \leq 1 + \dim(\mathfrak{a} + \mathfrak{b})$.

Hint 1*: Show that the general case can be reduced to the case in which*

$$\dim(\mathfrak{a} - \mathfrak{p}) < \dim \mathfrak{a} \text{ for all } \mathfrak{p} \in \mathbb{P}_F.$$

Hint 2: *For the reduced case, show that there exists* $0 \neq z \in \mathcal{L}(\mathfrak{a}) \setminus \bigcup_{\mathfrak{p} \in \mathrm{supp}(\mathfrak{b})} \mathcal{L}(\mathfrak{a} - \mathfrak{p})$. *Consider the map* $\varphi \colon \mathcal{L}(\mathfrak{b}) \to \mathcal{L}(\mathfrak{a} + \mathfrak{b})/\mathcal{L}(\mathfrak{a})$ *given by*

$$\varphi(x) = [z \cdot x] \mod \mathcal{L}(\mathfrak{a}).$$

What can you say about its kernel?

(b) Conclude that the same holds for all $\mathfrak{a}, \mathfrak{b}$ *with* $\dim \mathfrak{a}, \dim \mathfrak{b} > 0$.