Constant Field Extensions

Unit 19

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May 8, 2022

Constant field extensions

Definition 1 (Constant field extensions)

Let E/K be a function field, and L a field extension of K. Denote

$$F = LE.$$

Assuming F/L is a function field we say that F/L is a constant field extension of E/K.

It is not generally true that F/L as defined above is a function field, even assuming L/K is finite. But, as we will show, this is the case if L/K is separable.

Recall that if K is a finite field and L/K is finite, then L/K is separable. Indeed, finite fields are perfect.

Overview

Normal closure

Definition 2

Let M/K be a field extension, and fix an algebraic closure $\bar{\mathsf{K}}$ of K containing M. The normal closure $\widehat{\mathsf{M}}$ of M/K is the smallest subfield of $\bar{\mathsf{K}}$ containing M that is normal over K.

Assuming M/K is finite we can write

$$\mathsf{M} = \mathsf{K}(\alpha_1, \ldots, \alpha_n)$$

for some $\alpha_1, \ldots, \alpha_n \in \mathsf{M}$.

Let $\sigma_1, \ldots, \sigma_k$ be the embeddings of M in $\bar{\mathsf{K}}$. Then,

$$\widehat{\mathsf{M}} = \mathsf{K}\left(\{\sigma_i(\alpha_j) \mid i \in [k],\ j \in [n]\}\right).$$

In particular, when M/K is finite so is $\widehat{\mathsf{M}}/\mathsf{K}$.

Normal closure

$$\widehat{\mathsf{M}} = \mathsf{K}\left(\{\sigma_i(\alpha_j) \mid i \in [k],\ j \in [n]\}\right).$$

Note further that if M/K is separable then $\widehat{\mathsf{M}}/\mathsf{K}$ is Galois. Indeed,

$$\mathsf{M}/\mathsf{K} \text{ is separable} \quad \Longrightarrow \quad \widehat{\mathsf{M}}/\mathsf{K} \text{ is separable}$$

as we only adjoined K-conjugates of elements that are separable over K.

We conclude that $\widehat{\mathsf{M}}/\mathsf{K}$ is Galois as it is also normal. In this case we call $\widehat{\mathsf{M}}$ the Galois closure of M/K.

Compositum of purely inseparable extensions

Lemma 3

Let M/K *be a field extension and*

$$\mathrm{K} \subseteq \mathrm{E}_1, \mathrm{E}_2 \subseteq \mathrm{M},$$

where $\mathrm{E}_1, \mathrm{E}_2$ *are purely inseparable over* K. *Then,* $\mathrm{E}_1\mathrm{E}_2$ *is purely inseparable over* K.

Proof.

The set of purely inseparable elements in a field extension is an intermediate field. Indeed, recall that a is purely inseparable over K iff $a^{p^{e_a}} \in \mathrm{K}$ for some integer $e_a \geq 0$. So, if a, b are purely inseparable over K then, for $e = \max(e_a, e_b)$,

$$(a+b)^{p^e} = a^{p^e} + b^{p^e} \in \mathrm{F}.$$

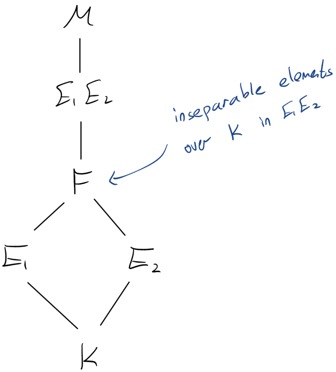
Same for multiplication and inverse.

Compositum of purely inseparable extensions

Proof.

Let F be the field of inseparable elements in E_1E_2 over K. Clearly, $E_1 \subseteq F$ and $E_2 \subseteq F$.

But, E_1E_2 is the smallest field containing E_1, E_2 and so $E_1E_2 = F$. Namely, all elements in E_1E_2 are purely inseparable over K.



The primitive element theorem

Recall

Theorem 4 (The primitive element theorem)

Every finite separable extension is simple.

Steinitz established the following generalization (note that finite fields are handled differently.)

Theorem 5

Assume K *is an infinite field. A finite field extension* M/K *is simple iff there are finitely many intermediate fields* $K \subseteq E \subseteq M$.

The latter implies the former as follows: Let M/K be a finite separable extension, and consider the normal closure $\widehat{M}$ of M/K. Then, $\widehat{M}/K$ is a finite Galois extension. By Galois Theory, there is a finite number of intermediate fields in $\widehat{M}/K$.

A variant of the primitive element theorem

Lemma 6

Let M/K *be a finite field extension, and denote* $p = \operatorname{char}\mathrm{K}$*. Then,*

$$[\mathrm{M}:\mathrm{K}]_i = p \quad \Longrightarrow \quad \mathrm{M}/\mathrm{K} \text{ is simple.}$$

Note that the primitive element theorem concludes the same under the assumption $[\mathrm{M}:\mathrm{K}]_i = 1$.

Proof.

The assertion is trivial for a finite field K.

Using Steinitz' Theorem (Theorem 5), it suffices to prove that M/K has finitely many intermediate fields.

A variant of the primitive element theorem

Proof.

We first show that K has at most one purely inseparable extension

$$\mathsf{K} \subsetneq \mathsf{E} \subseteq \mathsf{M}.$$

Indeed, assume two such extensions $\mathsf{E}_1, \mathsf{E}_2$ exist. Then,

$$[\mathsf{E}_1 : \mathsf{K}] = [\mathsf{E}_1 : \mathsf{K}]_i \geq p$$

(as it is a power of p and $\mathsf{E}_1 \neq \mathsf{K}$). Thus, since $\mathsf{E}_2 \neq \mathsf{E}_1$,

$$[\mathsf{E}_1\mathsf{E}_2 : \mathsf{K}] > p.$$

But by Lemma 3, $\mathsf{E}_1\mathsf{E}_2$ is a purely inseparable extension of K, and so

$$p = [\mathsf{M} : \mathsf{K}]_i \geq [\mathsf{E}_1\mathsf{E}_2 : \mathsf{K}]_i = [\mathsf{E}_1\mathsf{E}_2 : \mathsf{K}] > p,$$

which is a contradiction.

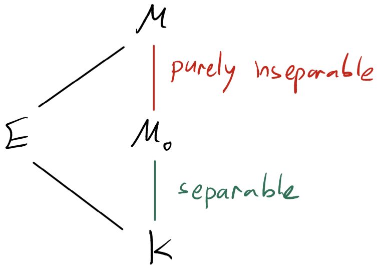
A variant of the primitive element theorem

Proof.

Let M_0 be the separable closure of K in M.

Consider now an intermediate field $K \subseteq E \subseteq M$.

Further consider the separable closure of K in E.



A variant of the primitive element theorem

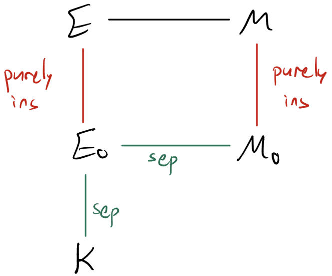
Proof.

We have that

$$[\mathrm{M} : \mathrm{E}_0]_i = \frac{[\mathrm{M} : \mathrm{K}]_i}{[\mathrm{E}_0 : \mathrm{K}]_i} = \frac{p}{1} = p.$$

Thus, E_0 has at most one purely inseparable extension in M (other than E_0), and so we can identify E with E_0.

But M_0/K is finite and separable, and so it has only finitely many intermediate fields E_0, which completes the proof. □



Overview

Separable constant field extensions

Theorem 7

Let E/K *be a function field and* L/K *finite and separable. Then,* LE/L *is a function field.*

Proof.

It suffices to prove that for every non-trivial finite extension M/L (namely, $M \neq L$) it holds that $M \not\subseteq LE$.

Indeed, if $\alpha \in LE \setminus L$ is algebraic over L then we can take

$$M = L(\alpha).$$

M/L is finite and $M \neq L$. But, of course, $M \subseteq LE$, in contradiction.

Moreover, we can assume that M/L has no intermediate fields as otherwise we can descend to one.

In particular, M/L is either separable or purely inseparable.

Separable constant field extensions

Proof.

In the second case, we may assume that

$$[\mathsf{M} : \mathsf{L}]_i = p.$$

Indeed,

$$[\mathsf{M} : \mathsf{L}] = [\mathsf{M} : \mathsf{L}]_i = p^e$$

for some $e \geq 1$. Take $\alpha \in \mathsf{M}$. If $[\mathsf{L}(\alpha) : \mathsf{L}] < p^e$ we can descend to $\mathsf{L}(\alpha)$. Otherwise, the minimal polynomial of α is

$$(T - \alpha)^{p^e} = T^{p^e} - \alpha^{p^e} \in \mathsf{L}[T].$$

Consider $\beta = \alpha^p$. Note that $(T - \beta)^{p^{e-1}} \in \mathsf{L}[T]$ vanishes at β and so

$$[\mathsf{L}(\beta) : \mathsf{L}] \leq p^{e-1}.$$

Thus, we can take $\mathsf{L}(\beta)$ instead of M and proceed this way until we get a degree p inseparable extension of L.

Separable constant field extensions

Proof.

In any case,

$$[\mathsf{M} : \mathsf{L}]_i \in \{1, p\},$$

and so, since L/K is separable,

$$[\mathsf{M} : \mathsf{K}]_i = [\mathsf{M} : \mathsf{L}]_i \cdot [\mathsf{L} : \mathsf{K}]_i \in \{1, p\}.$$

The primitive element theorem and its extension given by Lemma 6 imply that M/K is simple, namely, for some $\beta \in \mathsf{M}$,

$$\mathsf{M} = \mathsf{K}(\beta).$$

$$\begin{tikzcd}
E \arrow[r, dash] \arrow[d, dash] & LE \arrow[r, dash] \arrow[d, dash] & ME \arrow[d, dash] \\
K \arrow[r, dash] & L \arrow[r, dash] & M = K(\beta)
\end{tikzcd}$$

Separable constant field extensions

Recall a claim we proved when we talked about normal extensions.

Claim 8

Let E/K be a field extension s.t. K is algebraically closed in E. Then,

$$f \in \mathsf{K}[x] \text{ is irreducible} \quad \Longrightarrow \quad f \text{ is irreducible in } \mathsf{E}[x].$$

With this we return to the proof of Theorem 7.

Proof. (Proof of Theorem 7)

Let $f(x) \in \mathsf{K}[x]$ be the minimal polynomial of β over K. Then, by Claim 8, $f(x)$ is also the minimal polynomial of β over E.

$$\begin{tikzcd}
E \arrow[r, dash] \arrow[d, dash] & LE \arrow[r, dash] \arrow[d, dash] & ME \arrow[d, dash] \\
K \arrow[r, dash] & L \arrow[r, dash] & M = K(\beta)
\end{tikzcd}$$

Separable constant field extensions

Proof.

Let $f(x) \in \mathsf{K}[x]$ be the minimal polynomial of β over K. Then, $f(x)$ is also the minimal polynomial of β over E. Thus,

$$\mathsf{ME} = \mathsf{EK}(\beta) = \mathsf{E}(\beta),$$

and

$$[\mathsf{ME} : \mathsf{E}] = \deg f = [\mathsf{M} : \mathsf{K}].$$

But, $\mathsf{M} \neq \mathsf{L}$ and so

$$[\mathsf{ME} : \mathsf{E}] = [\mathsf{M} : \mathsf{K}] > [\mathsf{L} : \mathsf{K}] \geq [\mathsf{LE} : \mathsf{E}].$$

Thus, $\mathsf{ME} \neq \mathsf{LE}$, and so $\mathsf{M} \not\subseteq \mathsf{LE}$, as desired.

$$\begin{tikzcd}
E \arrow[r, dash] \arrow[d, dash] & LE \arrow[r, dash] \arrow[d, dash] & ME = E(\beta) \arrow[d, dash] \\
K \arrow[r, dash] & L \arrow[r, dash] & M = K(\beta)
\end{tikzcd}$$

Separable constant field extensions

From this point on up until the last part of this unit,

$$F/L = LE/L$$

refers to a constant field extension of E/K where L is a finite separable extension of K.

Separable constant field extensions

Lemma 9

Under the above,

$$[\mathsf{F} : \mathsf{E}] = [\mathsf{L} : \mathsf{K}].$$

Moreover, $\forall \mathfrak{a} \in \mathcal{D}(\mathsf{E}/\mathsf{K})$ *it holds that*

$$\deg_{\mathsf{F}} \mathfrak{a} = \deg_{\mathsf{E}} \mathfrak{a}.$$

Proof.

Since L/K is separable, $\mathsf{L} = \mathsf{K}(\alpha)$ for some $\alpha \in \mathsf{L}$. Thus,

$$\mathsf{F} = \mathsf{LE} = \mathsf{E}(\alpha).$$

Let $f(x) \in \mathsf{K}[x]$ be the minimal polynomial of α over K. By Claim 8, $f(x)$ is irreducible over E, and so

$$[\mathsf{F} : \mathsf{E}] = [\mathsf{E}(\alpha) : \mathsf{E}] = \deg f = [\mathsf{K}(\alpha) : \mathsf{K}] = [\mathsf{L} : \mathsf{K}].$$

Separable constant field extensions

Proof.

Per our assumption that L/K is finite we have that F/E is finite.

We proved in the previous unit that for a finite extension F/E,

$$\deg_F \mathfrak{a} = \frac{[F:E]}{[L:K]} \cdot \deg_E \mathfrak{a}.$$

Thus, in our case,

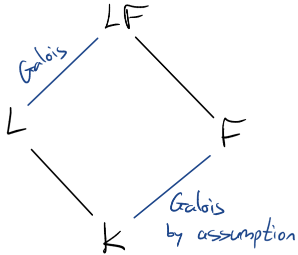
$$\deg_F \mathfrak{a} = \deg_E \mathfrak{a}.$$

Overview

A lemma from Galois Theory

Lemma 10

Let $\mathsf{K} \subseteq \mathsf{L}, \mathsf{F}$ *be fields s.t.* F/K *is Galois. Then* LF/L *is Galois.*



Proof.

The separability of LF/L is clear. Indeed, every element of F is separable over K, let alone over L. Thus, every element of LF is separable over L.

We turn to prove normality.

A lemma from Galois Theory

Proof.

As for normality, recall the characterization of normal extensions as splitting fields.

As F/K is normal, F is the splitting field of

$$\{f_j(x) \in \mathsf{K}[x]\}_{j \in J}.$$

Let $S_j \subseteq \mathsf{K}$ be the roots of $f_j(x)$, and $S = \cup_j S_j$. Then, $\mathsf{F} = \mathsf{K}(S)$. But then

$$\mathsf{LF} = \mathsf{LK}(S) = \mathsf{L}(S)$$

is the splitting field of

$$\{f_j(x) \in \mathsf{L}[x]\}_{j \in J}.$$

Hence, LF/L is normal. □

Residue fields under constant field extensions

Theorem 11

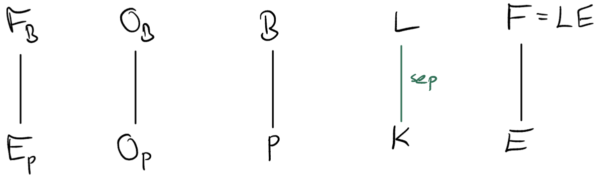
Let $\mathfrak{P}$ be a prime divisor of F/L *lying over* $\mathfrak{p}$ *in* E/K. *Then,*

$$F_{\mathfrak{P}} = (LE)_{\mathfrak{P}} = LE_{\mathfrak{p}}.$$

Proof.

The $\supseteq$ direction follows as both $L, E_{\mathfrak{p}} \subseteq F_{\mathfrak{P}}$.

Take $\bar{z} \in F_{\mathfrak{P}}$. We want to show $\bar{z} \in LE_{\mathfrak{p}}$.



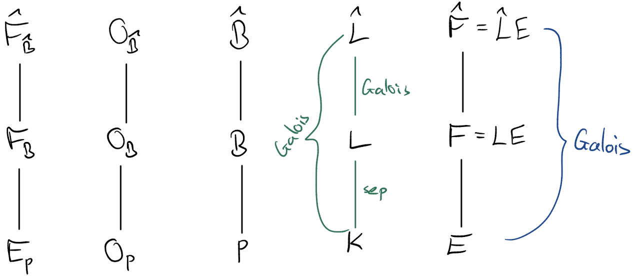
Residue fields under constant field extensions

Proof.

Let $\widehat{L}$ be the normal closure of L/K. Recall that $\widehat{L}/K$ is not only normal but also separable since L/K is separable. So $\widehat{L}/K$ is Galois.

By Lemma 10,

$$\widehat{L}/K \text{ is Galois} \quad \Longrightarrow \quad (\widehat{L}E)/(KE) = \widehat{F}/E \text{ is Galois.}$$



Residue fields under constant field extensions

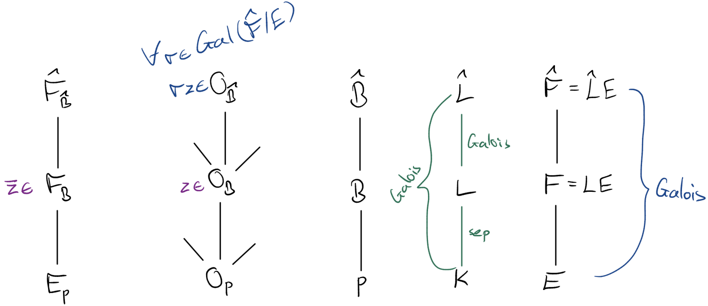
Proof.

Fix $\widehat{\mathfrak{P}}/\mathfrak{P}$. We turn to prove that $\bar{z}$ has a representative $z \in \mathcal{O}_{\mathfrak{P}}$ s.t.

$$\forall \sigma \in \mathsf{Gal}(\widehat{\mathsf{F}}/\mathsf{E}) \qquad \sigma z \in \mathcal{O}_{\widehat{\mathfrak{P}}}.$$

Let $z' \in \mathcal{O}_{\mathfrak{P}}$ that represents $\bar{z}$. By the WAT, $\exists z \in \mathsf{F}$ s.t.

$$v_{\mathfrak{P}}(z - z') > 0 \quad \text{and} \quad \forall \mathfrak{P}'/\mathfrak{p} \in \mathbb{P}_{\mathsf{F}} \setminus \{\mathfrak{P}\} \quad v_{\mathfrak{P}'}(z) \geq 0.$$



Residue fields under constant field extensions

Proof.

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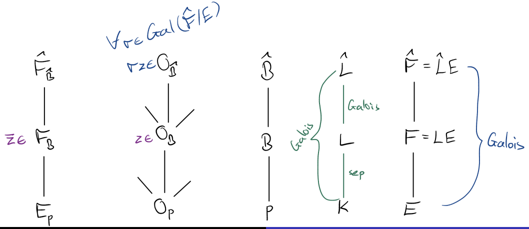
Thus, z is also a representative of $\bar{z}$. Moreover,

$$\forall \widehat{\mathfrak{P}}'/\mathfrak{p} \in \mathbb{P}_{\widehat{\mathsf{F}}} \qquad v_{\widehat{\mathfrak{P}}'}(z) \geq 0,$$

and so

$$\forall \sigma \in \mathrm{Gal}(\widehat{\mathsf{F}}/\mathsf{E}) \quad v_{\widehat{\mathfrak{P}}}(\sigma z) = v_{\sigma^{-1}\widehat{\mathfrak{P}}}(z) \geq 0 \quad \Longrightarrow \quad \sigma z \in \mathcal{O}_{\widehat{\mathfrak{P}}}.$$



Residue fields under constant field extensions

Proof.

Denote $n = [\mathsf{L} : \mathsf{K}]$ and let α be a primitive element of L/K. Then,

$$1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$$

is a basis for L/K.

By Lemma 9,

$$[\mathsf{F} : \mathsf{E}]_s = [\mathsf{F} : \mathsf{E}] = [\mathsf{L} : \mathsf{K}] = n,$$

so there are precisely n distinct embeddings of $\mathsf{F} \hookrightarrow \widehat{\mathsf{F}}$ over E which we denote by $\sigma_0, \ldots, \sigma_{n-1}$.

We have that

$$\mathsf{F} = \mathsf{LE} = \mathsf{E}(\alpha)$$

and so $1, \alpha, \ldots, \alpha^{n-1}$ is also a basis for F/E. Thus,

$$z = \sum_{j=0}^{n-1} x_j \alpha^j \qquad x_0, \ldots, x_{n-1} \in \mathsf{E}.$$

Residue fields under constant field extensions

Proof.

$$z = \sum_{j=0}^{n-1} x_j \alpha^j \qquad x_0, \ldots, x_{n-1} \in \mathsf{E}.$$

Thus, for $i = 0, 1, \ldots, n-1$,

$$\sigma_i z = \sum_{j=0}^{n-1} x_j (\sigma_i \alpha)^j.$$

Thus we are looking at a linear system of n equations in n unknowns $x_0, \ldots, x_{n-1}$. The corresponding matrix A satisfies

$$A_{i,j} = (\sigma_i \alpha)^j.$$

Observe that A is a matrix over $\widehat{\mathsf{L}}$. Indeed, $\alpha \in \mathsf{L}$, $(\sigma_i)|_{\mathsf{K}} = \mathrm{id}_{\mathsf{K}}$ and so $(\sigma_i)|_{\mathsf{L}} : \mathsf{L} \to \widehat{\mathsf{L}}$. Thus, $\sigma_i \alpha \in \widehat{\mathsf{L}}$.

Residue fields under constant field extensions

Proof.

$$A_{i,j} = (\sigma_i \alpha)^j \in \widehat{\mathsf{L}}.$$

Note that A is a Vandermonde matrix and so

$$\det A = \prod_{j<\ell} (\sigma_\ell \alpha - \sigma_j \alpha).$$

Since L/K is separable, $\sigma_\ell \alpha \neq \sigma_j \alpha$ for $j \neq \ell$. Indeed, otherwise σ_ℓ and σ_j will be equal on $\mathsf{K}(\alpha) = \mathsf{L}$.

Thus, $\det A \neq 0$ and so A has an inverse B over $\widehat{\mathsf{L}}$. So,

$$\forall 0 \leq j < n \quad x_j = \sum_{i=0}^{n-1} b_{ji} \sigma_i z \in \mathrm{Span}_{\widehat{\mathsf{L}}}(\sigma_0 z, \ldots, \sigma_{n-1} z).$$

But, recall that $x_j \in \mathsf{E}$.

Proof.

$$\forall j \quad x_j \in \mathsf{E} \cap \mathsf{Span}_{\widehat{\mathsf{L}}}(\sigma_0 z, \ldots, \sigma_{n-1} z).$$

Recap. recall that we took $\bar{z} \in \mathsf{F}_{\mathfrak{P}}$ and we wish to prove that $\bar{z} \in \mathsf{LE}_{\mathfrak{p}}$. We further took a representative $z \in \mathcal{O}_{\mathfrak{P}}$ s.t.

$$\sigma_0 z, \ldots, \sigma_{n-1} z \in \mathcal{O}_{\widehat{\mathfrak{P}}}.$$

But $\widehat{\mathsf{L}} \subseteq \mathcal{O}_{\widehat{\mathfrak{P}}}$ and so

$$x_0, \ldots, x_{n-1} \in \mathsf{E} \cap \mathcal{O}_{\widehat{\mathfrak{P}}} = \mathcal{O}_{\mathfrak{p}}.$$

Recall that $z = \sum_{j=0}^{n-1} x_j \alpha^j$ and so

$$\bar{z} = \sum_{j=0}^{n-1} \bar{x}_j \alpha^j \in \mathsf{E}_{\mathfrak{p}} \mathsf{L}.$$

$\square$

Overview

Riemann-Roch spaces in separable constant field extensions

We recall the following basic fact from Galois Theory.

Claim 12

Let $\mathrm{M}_1, \mathrm{M}_2$ be two field extensions of a field K, and assume M extend both M_1 and M_2.

Assume that every set $A_1 \subseteq \mathrm{M}_1$ that is linearly independent over K is also linearly independent over M_2 (were we think of $A_1 \subseteq \mathrm{M}$).

Then, every set $A_2 \subseteq \mathrm{M}_2$ that is linearly independent over K is also linearly independent over M_1 (where we think of $A_2 \subseteq \mathrm{M}$).

$$\begin{tikzcd}
M_2 \arrow[r, dash] & M \\
K \arrow[u, dash] \arrow[r, dash] & M_1 \arrow[u, dash]
\end{tikzcd}$$

Proof.

Let $y_1, \dots, y_n \in \mathrm{M}_2$ be linearly independent over K. Take $\alpha_1, \dots, \alpha_n \in \mathrm{M}_1$ s.t.

$$\sum_{i=1}^{n} \alpha_i y_i = 0.$$

We wish to show that $\alpha_1 = \cdots = \alpha_n = 0$.

Let $x_1, \dots, x_m \in \mathrm{M}_1$ be a basis of

$$\mathrm{Span}_{\mathrm{K}}(\alpha_1, \dots, \alpha_n) \subseteq \mathrm{M}_1.$$

Then, for every $i \in [n]$,

$$\alpha_i = \sum_{k=1}^{m} b_{ik} x_k,$$

with $\{b_{ik}\}_{i,k} \subseteq \mathrm{K}$.

Riemann-Roch spaces in separable constant field extensions

Proof.

$$\sum_{i=1}^{n} \alpha_i y_i = 0 \qquad \alpha_i = \sum_{k=1}^{m} b_{ik} x_k.$$

So,

$$0 = \sum_{i=1}^{n} \left(\sum_{k=1}^{m} b_{ik} x_k \right) y_i = \sum_{k=1}^{m} \left(\sum_{i=1}^{n} b_{ik} y_i \right) x_k.$$

Recall that Let $x_1, \ldots, x_m \in \mathrm{M}_1$ are linearly independent over K. Thus, per our assumption they are also linearly independent over M_2.

Recall that $\{b_{ik}\}_{i,k} \subseteq \mathrm{K}$ and that $y_1, \ldots, y_n \subseteq \mathrm{M}_2$. Thus, for every $k \in [m]$,

$$\sum_{i=1}^{n} b_{ik} y_i = 0.$$

But $y_1, \ldots, y_n$ are linearly independent over K and so $b_{ik} = 0$, and so $\alpha_1 = \cdots = \alpha_n = 0$.

Riemann-Roch spaces in separable constant field extensions

Corollary 13

Assume F/L *is a constant field extension of* E/K *with* L/K *finite and separable.*

If $A \subseteq \mathsf{E}$ *is linearly independent over* K *then (viewed as a subset of* F*)* A *is linearly independent over* L*.*

Proof.

By Claim 12, it suffices to prove that if $A \subseteq \mathsf{L}$ is linearly independent over K then A is also linearly independent over E.

$$\begin{array}{ccc} E & \text{———} & F = LE \\ | & & | \\ K & \text{———} & L \end{array}$$

Riemann-Roch spaces in separable constant field extensions

Proof.

Since L/K is separable and finite, $\mathsf{L} = \mathsf{K}(\alpha)$ for some $\alpha \in \mathsf{L}$.

Denote $n = [\mathsf{L} : \mathsf{K}]$. Then $1, \alpha, \ldots, \alpha^{n-1}$ is a basis for L/K. Now,

$$\mathsf{F} = \mathsf{EL} = \mathsf{E}(\alpha)$$

and, as we proved, the minimal polynomial f of α over K is also its minimal polynomial over E, and so

$$[\mathsf{F} : \mathsf{E}] = \deg f = [\mathsf{L} : \mathsf{K}] = n.$$

Thus, $1, \alpha, \ldots, \alpha^{n-1}$ is also a basis for F/E.

Riemann-Roch spaces in separable constant field extensions

Proof.

Now take $\beta_1, \ldots, \beta_m \in \mathsf{L}$ that are linearly independent over K.

Complete this to a basis $\beta_1, \ldots, \beta_n$ of L/K.

Then, there is an invertible matrix M over K that changes bases from $\beta_1, \ldots, \beta_n$ to $1, \alpha, \ldots, \alpha^{n-1}$.

M is also invertible as a matrix over E and so, since $1, \alpha, \ldots, \alpha^{n-1}$ is a basis of F/E then $\beta_1, \ldots, \beta_n$ is also a basis of F/E.

In particular, $\beta_1, \ldots, \beta_m$ are linearly independent over E.

Riemann-Roch spaces in separable constant field extensions

Recall that if $\mathfrak{a}$ is a divisor of E/K then $\mathrm{Con}(\mathfrak{a})$ is the "respective" divisor in F/L. Indeed, for a prime divisor $\mathfrak{p} \in \mathsf{E}/\mathsf{L}$ we defined

$$\mathrm{Con}(\mathfrak{p}) = \sum_{\mathfrak{P}/\mathfrak{p}} e(\mathfrak{P}/\mathfrak{p})\mathfrak{P},$$

and $\mathrm{Con}(\mathfrak{a})$ was extended by linearity.

To keep notation simple, we denote $\mathrm{Con}(\mathfrak{a})$ by $\mathfrak{a}$ and infer from context. In particular, for a divisor $\mathfrak{a}$ of E/K, we use the convention

$$\mathcal{L}_{\mathsf{F}}(\mathfrak{a}) = \mathcal{L}_{\mathsf{F}}(\mathrm{Con}(\mathfrak{a})).$$

Theorem 14

Let $\mathfrak{a}$ be a divisor of E/K. Then,

$$\mathcal{L}_{\mathsf{F}}(\mathfrak{a}) = \mathsf{L}\mathcal{L}_{\mathsf{E}}(\mathfrak{a}) = \mathrm{Span}_{\mathsf{L}}(\mathcal{L}_{\mathsf{E}}(\mathfrak{a})).$$

Riemann-Roch spaces in separable constant field extensions

Proof.

We have that

$$\begin{aligned}\mathcal{L}_{\mathsf{E}}(\mathfrak{a}) &= \{x \in \mathsf{E} \mid (x) + \mathfrak{a} \geq 0\} \\ &\subseteq \{x \in \mathsf{F} \mid (x) + \mathfrak{a} \geq 0\} = \mathcal{L}_{\mathsf{F}}(\mathfrak{a}).\end{aligned}$$

Note that the more elaborated way of writing this is as follows:

$$\begin{aligned}\mathcal{L}_{\mathsf{E}}(\mathfrak{a}) &= \{x \in \mathsf{E} \mid (x) + \mathfrak{a} \geq 0\} \\ &= \{x \in \mathsf{E} \mid \mathrm{Con}((x) + \mathfrak{a}) \geq 0\} \\ &= \{x \in \mathsf{E} \mid \mathrm{Con}(x) + \mathrm{Con}\, \mathfrak{a} \geq 0\} \\ &\subseteq \{y \in \mathsf{F} \mid (y) + \mathrm{Con}\, \mathfrak{a} \geq 0\} = \mathcal{L}_{\mathsf{F}}(\mathfrak{a}).\end{aligned}$$

Anyhow, $\mathcal{L}_{\mathsf{E}}(\mathfrak{a}) \subseteq \mathcal{L}_{\mathsf{F}}(\mathfrak{a})$. But $\mathcal{L}_{\mathsf{F}}(\mathfrak{a})$ is an L-vector space, and so

$$\mathsf{L}\mathcal{L}_{\mathsf{E}}(\mathfrak{a}) \subseteq \mathcal{L}_{\mathsf{F}}(\mathfrak{a}).$$

Proof.

We turn to prove the other direction, namely, $\mathcal{L}_{\mathsf{F}}(\mathfrak{a}) \subseteq \mathsf{L}\mathcal{L}_{\mathsf{E}}(\mathfrak{a})$. To this end take $z \in \mathcal{L}_{\mathsf{F}}(\mathfrak{a})$ and consider again the Galois closure $\widehat{\mathsf{L}}$ of L/K. We turn to prove that

$$\forall \sigma \in \mathsf{Gal}(\widehat{\mathsf{F}}/\mathsf{E}) \quad \sigma z \in \mathcal{L}_{\widehat{\mathsf{F}}}(\mathfrak{a}).$$

$\forall \sigma \in Gal(\hat{F}/E)$

$\sigma z \in \mathcal{L}_{\hat{F}}(a)$ $\qquad$ $\hat{L}E = \hat{F}$ $\qquad$ $\hat{L}$

$z \in \mathcal{L}_F(a)$ $\qquad$ $LE = F$ $\qquad$ L

$\mathcal{L}_E(a)$ $\qquad$ E $\qquad$ K

$$\begin{tikzcd} \hat{L}E = \hat{F} \arrow[d, dash] \\ LE = F \arrow[d, dash] \\ E \end{tikzcd} \quad \text{Galois} \qquad\qquad \begin{tikzcd} \hat{L} \arrow[d, dash, "\text{Galois}"] \\ L \arrow[d, dash, "\text{sep}"] \\ K \end{tikzcd} \quad \text{Galois}$$

Riemann-Roch spaces in separable constant field extensions

Proof.

As $z \in \mathcal{L}_{\mathsf{F}}(\mathfrak{a})$,

$$(z) + \mathfrak{a} \geq 0$$

as divisors of $\widehat{\mathsf{F}}/\widehat{\mathsf{L}}$. Namely,

$$\forall \widehat{\mathfrak{P}} \in \mathbb{P}_{\widehat{\mathsf{F}}/\widehat{\mathsf{L}}} \quad v_{\widehat{\mathfrak{P}}}(z) + v_{\widehat{\mathfrak{P}}}(\mathfrak{a}) \geq 0.$$

In particular, for every such $\widehat{\mathfrak{P}}$,

$$v_{\sigma^{-1}\widehat{\mathfrak{P}}}(z) + v_{\sigma^{-1}\widehat{\mathfrak{P}}}(\mathfrak{a}) \geq 0.$$

But, $\widehat{\mathsf{F}}/\mathsf{E}$ is Galois, and so

$$\begin{aligned} v_{\widehat{\mathfrak{P}}}(\mathfrak{a}) &= e(\widehat{\mathfrak{P}}/\mathfrak{p}) v_{\mathfrak{p}}(\mathfrak{a}) \\ &= e(\sigma^{-1}\widehat{\mathfrak{P}}/\mathfrak{p}) v_{\mathfrak{p}}(\mathfrak{a}) \\ &= v_{\sigma^{-1}\widehat{\mathfrak{P}}}(\mathfrak{a}). \end{aligned}$$

Proof.

So far,

$$\forall \widehat{\mathfrak{P}} \quad v_{\sigma^{-1}\widehat{\mathfrak{P}}}(z) + v_{\sigma^{-1}\widehat{\mathfrak{P}}}(\mathfrak{a}) \geq 0.$$

and

$$v_{\widehat{\mathfrak{P}}}(\mathfrak{a}) = v_{\sigma^{-1}\widehat{\mathfrak{P}}}(\mathfrak{a}).$$

Thus,

$$\begin{aligned} v_{\widehat{\mathfrak{P}}}(\sigma z) + v_{\widehat{\mathfrak{P}}}(\mathfrak{a}) &= v_{\sigma^{-1}\widehat{\mathfrak{P}}}(z) + v_{\widehat{\mathfrak{P}}}(\mathfrak{a}) \\ &= v_{\sigma^{-1}\widehat{\mathfrak{P}}}(z) + v_{\sigma^{-1}\widehat{\mathfrak{P}}}(\mathfrak{a}) \geq 0. \end{aligned}$$

That is,

$$(\sigma z) + \mathfrak{a} \geq 0$$

as divisors of $\widehat{\mathsf{F}}/\widehat{\mathsf{L}}$, and so

$$\sigma z \in \mathcal{L}_{\widehat{\mathsf{F}}}(\mathfrak{a}).$$

Riemann-Roch spaces in separable constant field extensions

Proof.

By inspecting the proof of Theorem 11, we can write

$$z = \sum_{j=0}^{n-1} x_j \alpha^j$$

with

$$\begin{aligned} x_j &\in \mathsf{E} \cap \mathsf{Span}_{\widehat{\mathsf{L}}}\left(\left\{\sigma z \mid \sigma \in \mathsf{Gal}(\widehat{\mathsf{F}}/\mathsf{E})\right\}\right) \\ &\subseteq \mathsf{E} \cap \mathcal{L}_{\widehat{\mathsf{F}}}(\mathfrak{a}) \\ &= \mathcal{L}_{\mathsf{E}}(\mathfrak{a}). \end{aligned}$$

Therefore, as $\alpha \in \mathsf{L}$,

$$z \in \mathsf{L}\mathcal{L}_{\mathsf{E}}(\mathfrak{a}).$$

Riemann-Roch spaces in separable constant field extensions

Corollary 15

With the above notations,

$$\dim_F \mathfrak{a} = \dim_E \mathfrak{a}.$$

Proof.

By Theorem 14,

$$\mathcal{L}_F(\mathfrak{a}) = L\mathcal{L}_E(\mathfrak{a}) = \mathrm{Span}_L(\mathcal{L}_E(\mathfrak{a})).$$

Now,

$$\dim_F \mathfrak{a} = \dim_L \mathcal{L}_F(\mathfrak{a}) = \dim_L \mathrm{Span}_L(\mathcal{L}_E(\mathfrak{a})),$$
$$\dim_E \mathfrak{a} = \dim_K \mathcal{L}_E(\mathfrak{a}).$$

Thus, we need to show that

$$\dim_K \mathcal{L}_E(\mathfrak{a}) = \dim_L \mathrm{Span}_L(\mathcal{L}_E(\mathfrak{a})).$$

Riemann-Roch spaces in finite separable constant field extensions

Proof.

We need to show that

$$\dim_{\mathsf{K}} \mathcal{L}_{\mathsf{E}}(\mathfrak{a}) = \dim_{\mathsf{L}} \mathsf{Span}_{\mathsf{L}}(\mathcal{L}_{\mathsf{E}}(\mathfrak{a})).$$

Corollary 13 states that if $A \subseteq \mathsf{E}$ is linearly independent over K then (viewed as a subset of F) A is linearly independent over L. Taking A to be a basis of $\mathcal{L}_{\mathsf{E}}(\mathfrak{a})$ (over K) yields the $\leq$ direction.

The $\geq$ direction readily follows since there is a basis for $\mathsf{Span}_{\mathsf{L}}(\mathcal{L}_{\mathsf{E}}(\mathfrak{a}))$ (over L) that is contained in $\mathcal{L}_{\mathsf{E}}(\mathfrak{a})$. Such a basis certainly remains independent over K. □

Overview

The genus in finite separable constant field extensions

Theorem 16

If F/L *is a finite separable constant field extension of* E/K *and the respective genera are* $g_{\mathsf{F}}, g_{\mathsf{E}}$ *then*

$$g_{\mathsf{F}} = g_{\mathsf{E}}.$$

Proof.

Take $\mathfrak{p}$ a prime divisor of E/K and $k \in \mathbb{N}$ large enough so that

$$\min(\deg_{\mathsf{E}} \mathfrak{a}, \deg_{\mathsf{F}} \mathfrak{a}) \geq k \geq \max(2g_{\mathsf{E}} - 2, 2g_{\mathsf{F}} - 2),$$

where $\mathfrak{a} = k\mathfrak{p}$. By Riemann-Roch,

$$\dim_{\mathsf{E}} \mathfrak{a} = \deg_{\mathsf{E}} \mathfrak{a} + 1 - g_{\mathsf{E}},$$
$$\dim_{\mathsf{F}} \mathfrak{a} = \deg_{\mathsf{F}} \mathfrak{a} + 1 - g_{\mathsf{F}}.$$

The genus in finite separable constant field extensions

Proof.

$$\dim_E \mathfrak{a} = \deg_E \mathfrak{a} + 1 - g_E,$$
$$\dim_F \mathfrak{a} = \deg_F \mathfrak{a} + 1 - g_F.$$

Now, by Lemma 9,

$$\deg_E \mathfrak{a} = \deg_F \mathfrak{a},$$

and by Corollary 15,

$$\dim_E \mathfrak{a} = \dim_F \mathfrak{a}.$$

Therefore, $g_E = g_F$.

Overview

Characterization of constant field invariance

Lemma 17

Let E/K *a function field with* K *a perfect field. Let* L/K *be an algebraic extension (finite or infinite) and denote* $\mathsf{F} = \mathsf{EL}$*. Then,*

1. L *is algebraically closed in* F*.*
2. *Any subset of* E *that is* K*-linearly independent remains so over* L*.*
3. *For every* $x \in \mathsf{E} \setminus \mathsf{K}$*,*

$$[\mathsf{E} : \mathsf{K}(x)] = [\mathsf{F} : \mathsf{L}(x)].$$

Proof.

We start with Item 1. Take $\gamma \in \mathsf{F}$ that is algebraic over L. We wish to show $\gamma \in \mathsf{L}$. As $\mathsf{F} = \mathsf{EL}$,

$$\exists \alpha_1, \dots, \alpha_r \in \mathsf{L} \qquad \gamma \in \mathsf{E}(\alpha_1, \dots, \alpha_r).$$

Now $\mathsf{K}(\alpha_1, \dots, \alpha_r)/\mathsf{K}$ is finite hence separable, and so $\exists \alpha \in \mathsf{L}$ s.t $\mathsf{K}(\alpha_1, \dots, \alpha_r) = \mathsf{K}(\alpha)$.

Characterization of constant field invariance

Proof.

Recall that γ is algebraic over L and so it is algebraic over K. Indeed, consider the chain $L(\gamma)/L/K$. Thus, $K(\alpha,\gamma)/K$ is finite hence separable, and so

$$\exists \beta \in F \qquad K(\alpha,\gamma) = K(\beta).$$

Adjoining E we get that

$$E(\beta) = E(\alpha,\gamma) = E(\alpha),$$

where the last equality follows since $\gamma \in E(\alpha_1 \ldots, \alpha_r) = E(\alpha)$. Hence,

$$[K(\beta) : K] = \deg f_\beta = [E(\beta) : E] = [E(\alpha) : E] = \deg f_\alpha = [K(\alpha) : K].$$

Thus, $K(\alpha) = K(\beta)$ and so

$$\gamma \in K(\beta) = K(\alpha) \subseteq L.$$

Characterization of constant field invariance

Proof.

We turn to prove Item 2. Take $y_1, \ldots, y_r \in \mathsf{E}$ that are linearly independent over K. Assume that

$$\sum_{i=1}^{r} \gamma_i y_i = 0 \qquad \gamma_1, \ldots, \gamma_r \in \mathsf{L}.$$

We want to show that $\gamma_1 = \cdots = \gamma_r = 0$. Since K is perfect and $\mathsf{K}(\gamma_1, \ldots, \gamma_r)/\mathsf{K}$ is finite hence separable,

$$\exists \alpha \in \mathsf{L} \qquad \mathsf{K}(\gamma_1, \ldots, \gamma_r) = \mathsf{K}(\alpha).$$

For each $i \in [r]$, write

$$\gamma_i = \sum_{j=0}^{n-1} c_{i,j} \alpha^j \qquad c_{i,j} \in \mathsf{K},$$

where $n = [\mathsf{K}(\alpha) : \mathsf{K}] = \deg f_\alpha$.

Characterization of constant field invariance

Proof.

$$\sum_{i=1}^{r} \gamma_i y_i = 0 \qquad \gamma_1, \ldots, \gamma_r \in \mathsf{L},$$

$$\gamma_i = \sum_{j=0}^{n-1} c_{i,j} \alpha^j \qquad c_{i,j} \in \mathsf{K}.$$

So

$$0 = \sum_{i=1}^{r} \left(\sum_{j=0}^{n-1} c_{i,j} \alpha^j \right) y_i = \sum_{j=0}^{n-1} \left(\sum_{i=1}^{r} c_{i,j} y_i \right) \alpha^j .$$

Recall that $1, \alpha, \ldots, \alpha^{n-1}$ are linearly independent over E since K is algebraically closed in E.

Characterization of constant field invariance

Proof.

$$0 = \sum_{j=0}^{n-1} \left(\sum_{i=1}^{r} c_{i,j} y_i \right) \alpha^j .$$

$1, \alpha, \ldots, \alpha^{n-1}$ are linearly independent over E. Thus, for every j,

$$\sum_{i=1}^{r} c_{i,j} y_i = 0.$$

But $c_{i,j} \in$ K and $y_1, \ldots, y_r$ are linearly independent over K and so $c_{i,j} = 0$, and so are the γ_i-s.

Characterization of constant field invariance

Proof.

We turn to prove Item 3, namely,

$$\forall x \in \mathrm{E} \setminus \mathrm{K} \qquad [\mathrm{E} : \mathrm{K}(x)] = [\mathrm{F} : \mathrm{L}(x)].$$

The $\geq$ direction follows as we adjoin L to $\mathrm{E}/\mathrm{K}(x)$ and so the degree can only decrease.

As for the other direction, take $z_1, \ldots, z_s \in \mathrm{E}$ that are linearly independent over $\mathrm{K}(x)$. We wish to show these remain linearly independent over $\mathrm{L}(x)$. Otherwise,

$$\sum_{i=1}^{s} f_i(x) z_i = 0 \qquad f_i(x) \in \mathrm{L}[x],$$

where not all $f_i(x)$ zeros. Thus, $\{x^j z_i\}_{i,j}$ are linearly dependent over L, and so, by Item 2, also over K. Thus, $z_1, \ldots, z_s$ are linearly dependent over $\mathrm{K}(x)$ - a contradiction. $\square$

Characterization of constant field invariance

Theorem 18

Let F/L *be a finite function field extension of* E/K. *Assume* K *is a perfect field. Let* $\bar{\mathsf{K}}$ *be an algebraic closure of* K *(containing* L*). Then,*

$$[\mathsf{F} : \mathsf{E}] = [\mathsf{F}\bar{\mathsf{K}} : \mathsf{E}\bar{\mathsf{K}}] \cdot [\mathsf{L} : \mathsf{K}].$$

Proof.

First,

$$[\mathsf{F} : \mathsf{E}] = [\mathsf{F} : \mathsf{EL}] \cdot [\mathsf{EL} : \mathsf{E}].$$

Write $\mathsf{L} = \mathsf{K}(\alpha)$ and recall that $\mathsf{EL} = \mathsf{E}(\alpha)$ and that

$$[\mathsf{EL} : \mathsf{E}] = \deg f_\alpha = [\mathsf{L} : \mathsf{K}],$$

where f_α is the minimal polynomial of α over K.

So it remains to prove that

$$[\mathsf{F} : \mathsf{EL}] = [\mathsf{F}\bar{\mathsf{K}} : \mathsf{E}\bar{\mathsf{K}}].$$

Characterization of constant field invariance

Proof.

We wish to prove that

$$[\mathsf{F} : \mathsf{EL}] = [\mathsf{F}\bar{\mathsf{K}} : \mathsf{E}\bar{\mathsf{K}}].$$

Fix $x \in \mathsf{E} \setminus \mathsf{L}$. By Lemma 17 (taking the constant field extension $\mathsf{E}\bar{\mathsf{K}}/\bar{\mathsf{K}}$ of EL/L)

$$[\mathsf{EL} : \mathsf{L}(x)] = [\mathsf{E}\bar{\mathsf{K}} : \bar{\mathsf{K}}(x)].$$

Similarly, by considering the constant field extension $\mathsf{F}\bar{\mathsf{K}}/\bar{\mathsf{K}}$ of F/L,

$$[\mathsf{F} : \mathsf{L}(x)] = [\mathsf{F}\bar{\mathsf{K}} : \bar{\mathsf{K}}(x)].$$

Thus,

$$[\mathsf{F} : \mathsf{EL}] = \frac{[\mathsf{F} : \mathsf{L}(x)]}{[\mathsf{EL} : \mathsf{L}(x)]} = \frac{[\mathsf{F}\bar{\mathsf{K}} : \bar{\mathsf{K}}(x)]}{[\mathsf{E}\bar{\mathsf{K}} : \bar{\mathsf{K}}(x)]} = [\mathsf{F}\bar{\mathsf{K}} : \mathsf{E}\bar{\mathsf{K}}].$$

$\square$

Characterization of constant field invariance

Corollary 19

Let F/L *be a finite function field extension of* E/K*, with* K *perfect. Assume that* $\mathsf{F} = \mathsf{E}(y)$ *and that* $\varphi(T) \in \mathsf{E}[T]$ *is the minimal polynomial of* y *over* E*. Then, TFAE:*

1. $\mathsf{L} = \mathsf{K}$.
2. $\varphi(T)$ *is irreducible in* $\mathsf{E}\bar{\mathsf{K}}[T]$.

Proof.

By Theorem 18,

$$[\mathsf{F} : \mathsf{E}] = [\mathsf{F}\bar{\mathsf{K}} : \mathsf{E}\bar{\mathsf{K}}] \cdot [\mathsf{L} : \mathsf{K}],$$

and so (1) is equivalent to

$$[\mathsf{F} : \mathsf{E}] = [\mathsf{F}\bar{\mathsf{K}} : \mathsf{E}\bar{\mathsf{K}}].$$

Characterization of constant field invariance

Proof.

So far,

$$(1) \iff [F : E] = [F\bar{K} : E\bar{K}].$$

But $F = E(y)$ and so

$$[F : E] = [E(y) : E],$$
$$[F\bar{K} : E\bar{K}] = [E\bar{K}(y) : E\bar{K}].$$

So

$$(1) \iff [E(y) : E] = [E\bar{K}(y) : E\bar{K}].$$

The proof then follows since also

$$(2) \iff [E(y) : E] = [E\bar{K}(y) : E\bar{K}].$$

□

Characterization of constant field invariance

Corollary 20

Let F/K *be a finite extension of* E/K*, with* K *perfect. Then, for every algebraic separable extension* L/K*,*

$$[F : E] = [FL : EL].$$

Proof.

By Theorem 18,

$$[F : E] = [F\bar{K} : E\bar{K}] \cdot [K : K] = [F\bar{K} : E\bar{K}],$$

and (using that $\bar{K} = \bar{L}$),

$$[FL : EL] = [FL\bar{L} : EL\bar{L}] \cdot [L : L] = [F\bar{K} : E\bar{K}].$$

Therefore

$$[F : E] = [FL : EL].$$

□

Characterization of constant field invariance

Definition 21

A polynomial $\varphi(T) \in \mathsf{K}(x)[T]$ is said to be absolutely irreducible if $\varphi(T)$ is irreducible in $\bar{\mathsf{K}}(x)[T]$.

Theorem 22

Let F/K *be a field extension s.t.* $\mathsf{F} \neq \mathsf{K}$,

$$\mathsf{F} = \mathsf{K}(x, y),$$

and $[\mathsf{F} : \mathsf{K}(x)] < \infty$. *Assume* K *is perfect.*

Let $\varphi(T) \in \mathsf{K}(x)[T]$ *be the minimal polynomial of* y *over* $\mathsf{K}(x)$. *TFAE:*

1. F/K *is a function field;*
2. $\varphi(T)$ *is absolutely irreducible.*

Characterization of constant field invariance

Proof.

Per our assumption, $\mathsf{F} \neq \mathsf{K}$, $\mathsf{F} = \mathsf{K}(x, y)$ and $[\mathsf{F} : \mathsf{K}(x)] < \infty$. Thus, we need to prove that

$$\mathsf{K} \text{ is algebraically closed in } \mathsf{F} \quad \Longleftrightarrow \quad \varphi(T) \text{ is absolutely irreducible.}$$

Let L be the algebraic closure of K in F. Note that F/L is a function field. Indeed, $\mathsf{F} = \mathsf{L}(x, y) \neq \mathsf{L}$ (as F/K is of transcendence degree 1 and L/K is algebraic) and

$$[\mathsf{F} : \mathsf{L}(x)] \leq [\mathsf{F} : \mathsf{K}(x)] < \infty.$$

Moreover, L is algebraically closed in F (as the algebraic closure of K).

$$\begin{tikzcd}
& F = K(x,y) \arrow[dl, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & & K(x) \arrow[dl, dash] \\
& K &
\end{tikzcd}$$

Characterization of constant field invariance

Proof.

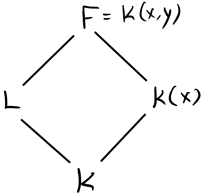
Consider the function field extension F/L over $K(x)/K$. This is a function field extension x is transcendental over K and so

$$L \cap K(x) = K.$$

Since $[F : K(x)] < \infty$, L/K is finite. Indeed,

$$[L : K] = [L(x) : K(x)] = \frac{[F : K(x)]}{[F : L(x)]}.$$

As K is perfect we conclude that L/K is separable.



Characterization of constant field invariance

Proof.

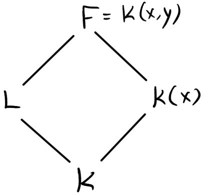
L/K is finite and separable, and so by Corollary 19 (with $E = K(x)$),

$$L = K \quad \Longleftrightarrow \quad \varphi(T) \text{ is irreducible in } K(x)\bar{K}[T].$$

The proof follows as

$$K(x)\bar{K} = \bar{K}(x).$$

$\square$



Example

Consider our running example $\mathsf{F} = \mathsf{K}(x, y)$ where K is a finite field and

$$y^2 = x^3 - x.$$

By Theorem 22, to prove that F/K is a function field, it suffices to prove that

$$T^2 - x^3 + x \in \bar{\mathsf{K}}(x)[T]$$

is irreducible.

If this is not the case then

$$T^2 - x^3 + x = (T + a(x))(T + b(x)),$$

with $a(x), b(x) \in \bar{\mathsf{K}}(x)$.

Example

$$T^2 - x^3 + x = (T + a(x))(T + b(x)),$$

with $a(x), b(x) \in \bar{K}(x)$.

But then $a(x) = -b(x)$ and

$$a(x)b(x) = x^3 - x,$$

and so

$$a(x)^2 = x - x^3$$

which forces $a(x) \in \bar{K}[x]$ and then yields a contradiction by degree considerations.