Winter School on expansion in groups, combinatorics, and complexity

Analytic Approaches to Spectral Graph Theory: Insights From Free Probability

Minicourse

Gil Cohen   Tel Aviv University

Plan.

* Spectral graph theory 101
* Expander graphs → Zig Zag Product

Reingold Vadhan Wigderson 2000

"Free" union of perfect matchings

Marcus Spielman Srivastava ... 2015

Fundamental questions we can't answer.

A detour to Free Probability Theory

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$\exists$ ONE more probability theory! Voiculescu 1985

* Freeness ≈ independence
* Central limit theorem
* Analytic machinery

II

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II

Finite FPT (interlacing, quadrature, ...)

* One-sided Ramanujan graphs
* Zig Zag revisited

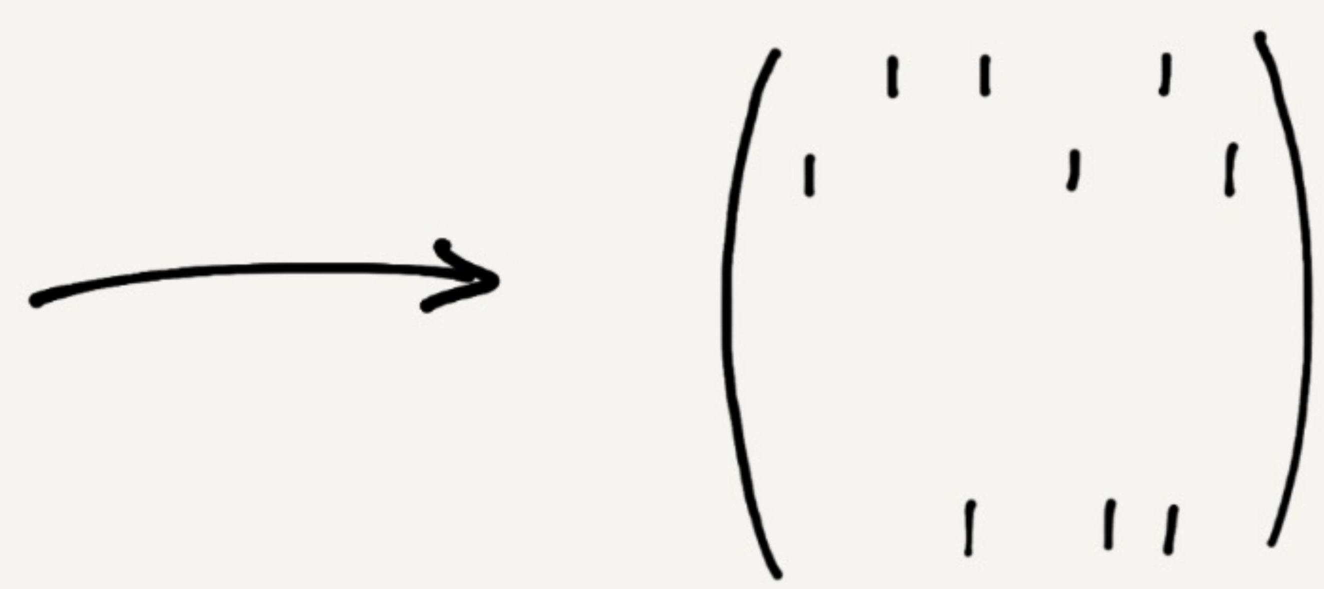
III

Spectral Graph Theory 101

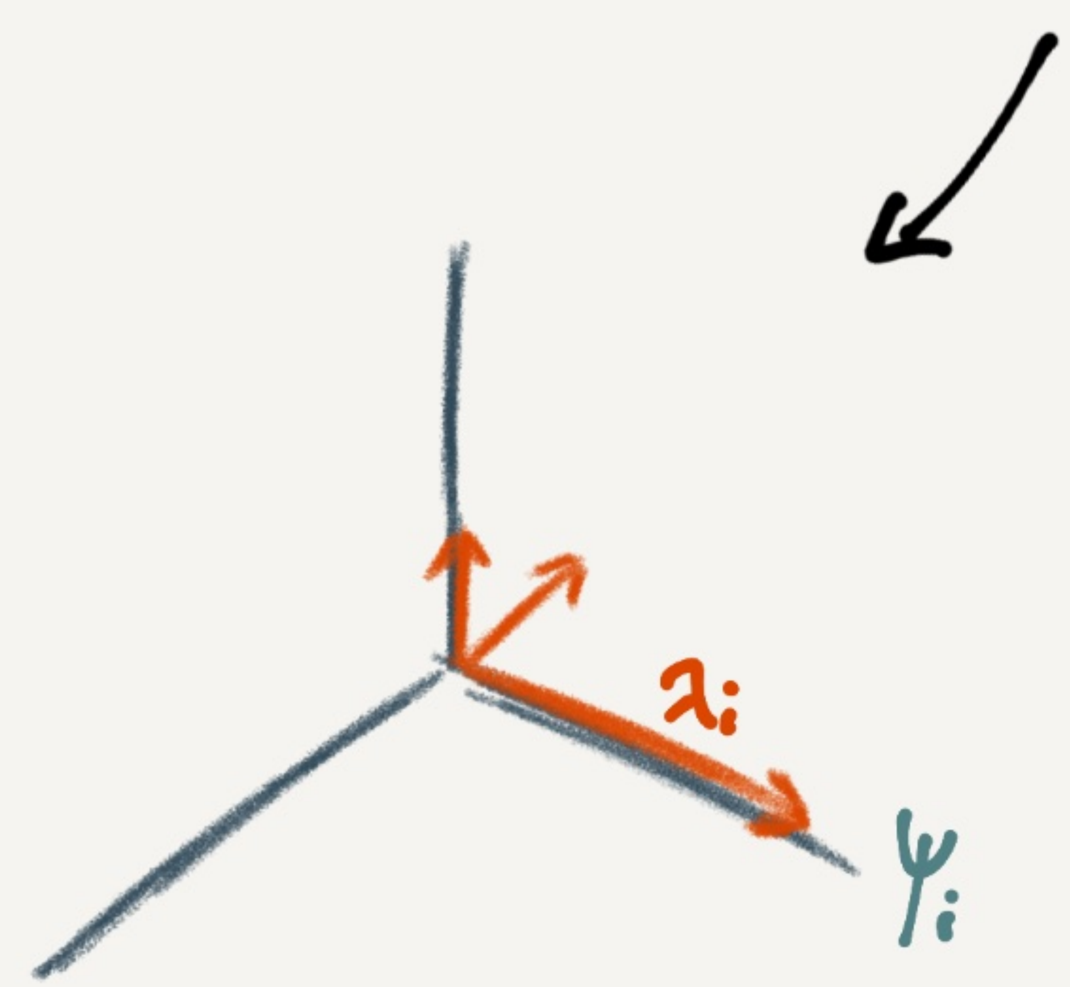
Spectral Graph Theory

The study of graphs through the lens of linear algebra.

G



A_G

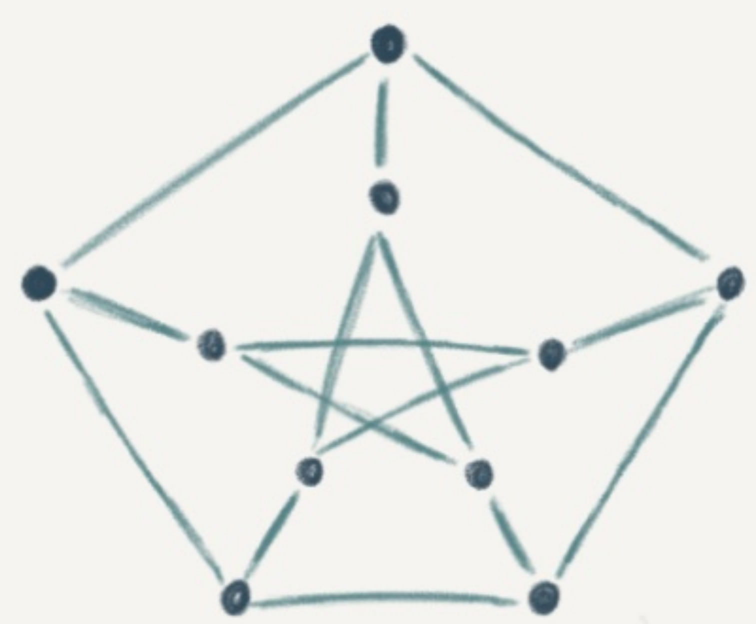


eigenvalue

$$A_G = \sum \lambda_i \psi_i \psi_i^T$$

eigenvector

$$\text{Spec } G = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$$



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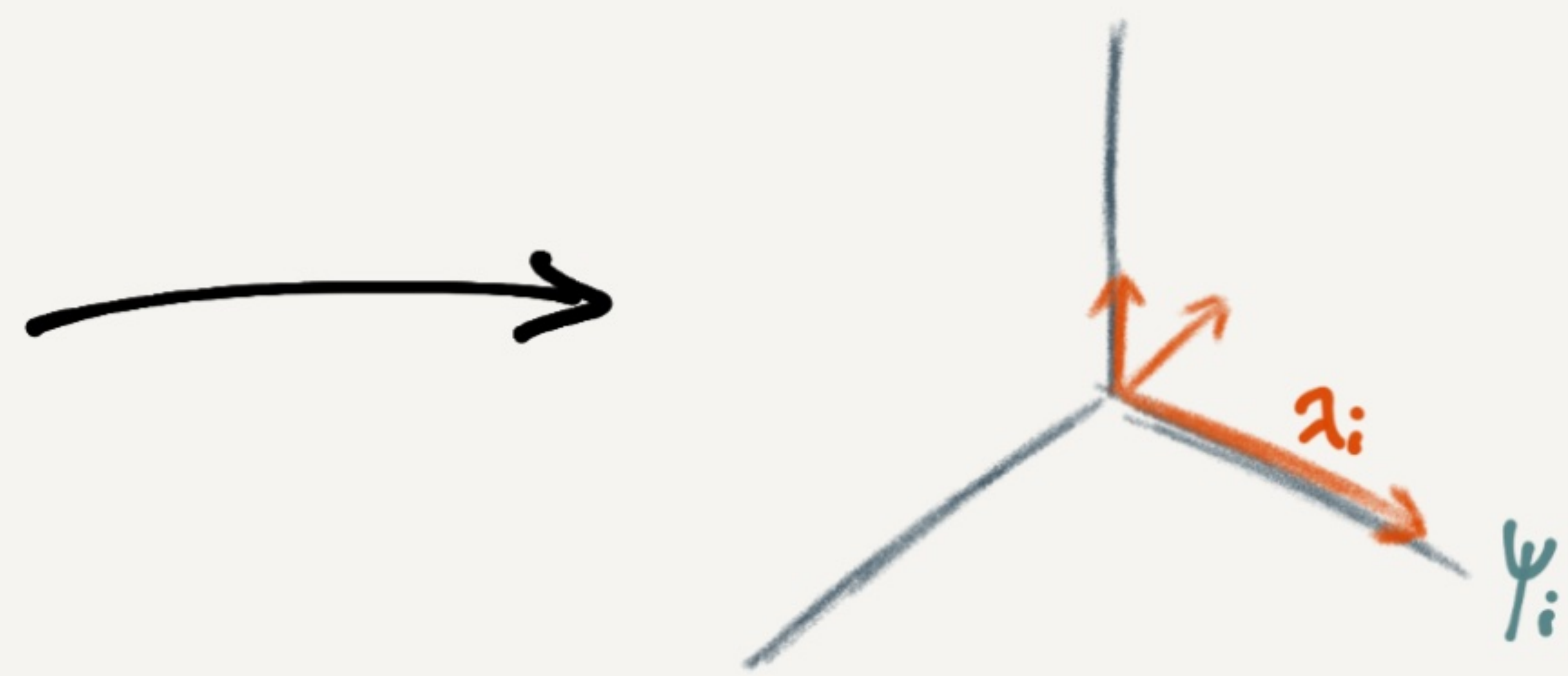
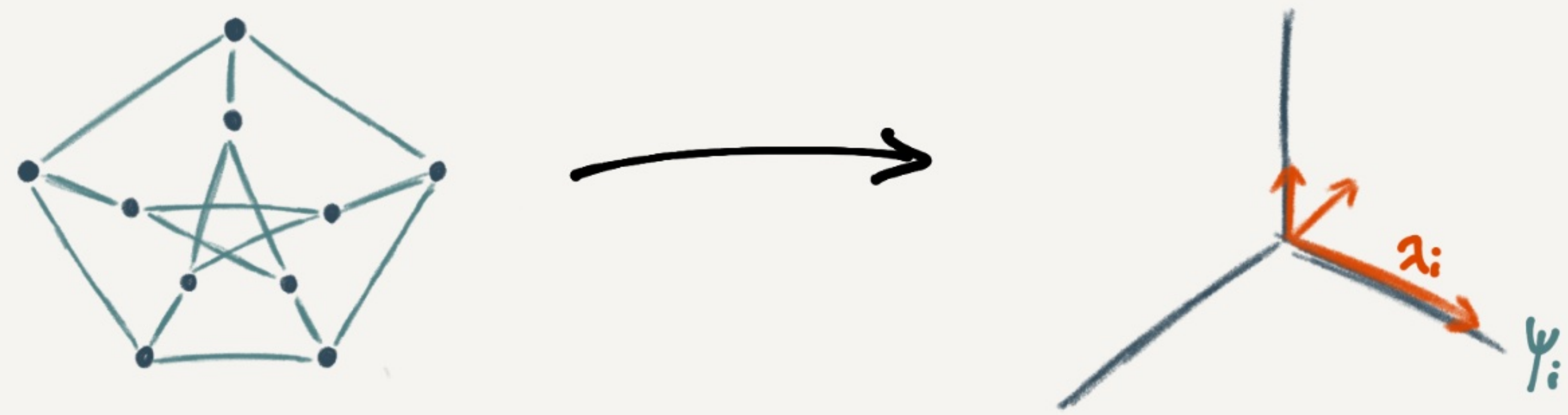


$$A_G = \sum \overset{\text{eigenvalue}}{\lambda_i}\, \underset{\text{eigenvector}}{\psi_i}\, \psi_i^T$$

$$\operatorname{Spec} G = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|}\}$$

G d-regular $\iff$ $\lambda_1 = d$



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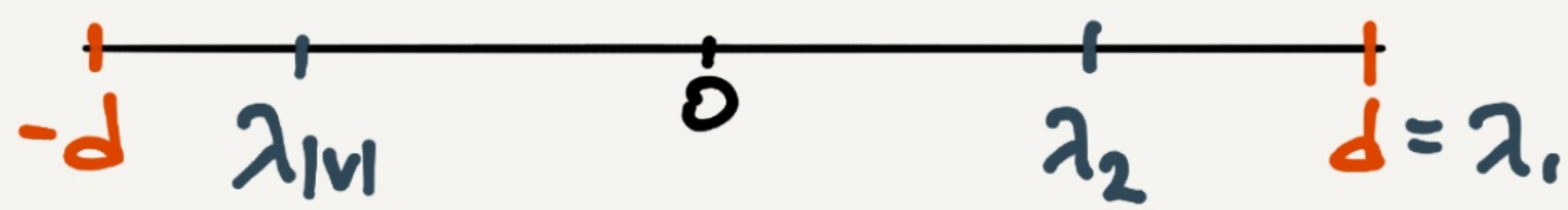

Def. $\lambda_G \triangleq \max\{\lambda_2, |\lambda_{|V|}|\}$ is called the spectral expansion of G.

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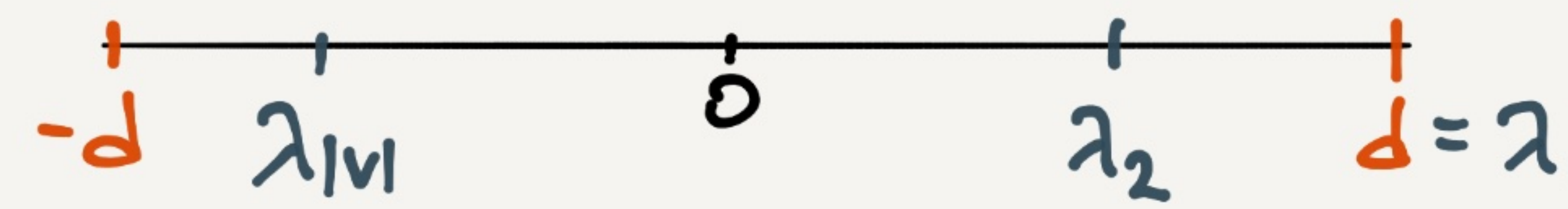

λ_G small $\Longrightarrow$ great things

Expander Chernoff bound

Random walks mix quickly

Expander mixing lemma

Thm [AB '91].

$\forall \varepsilon > 0$ only finitely many d-regular graphs G satisfy

$$\lambda_G < 2\sqrt{d-1} - \varepsilon$$

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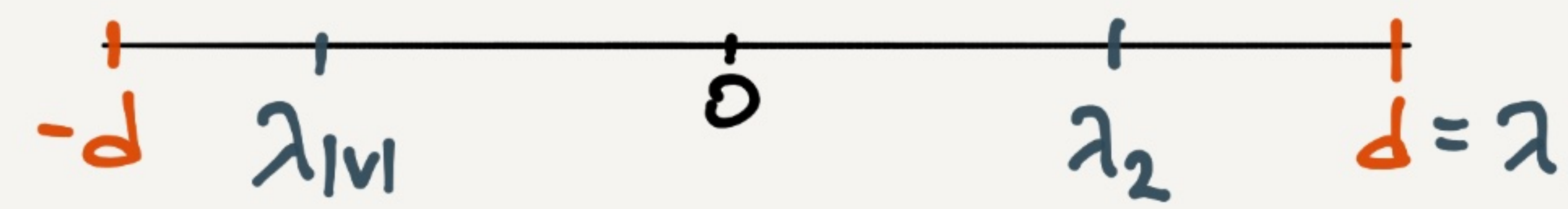


Thm [AB '91].

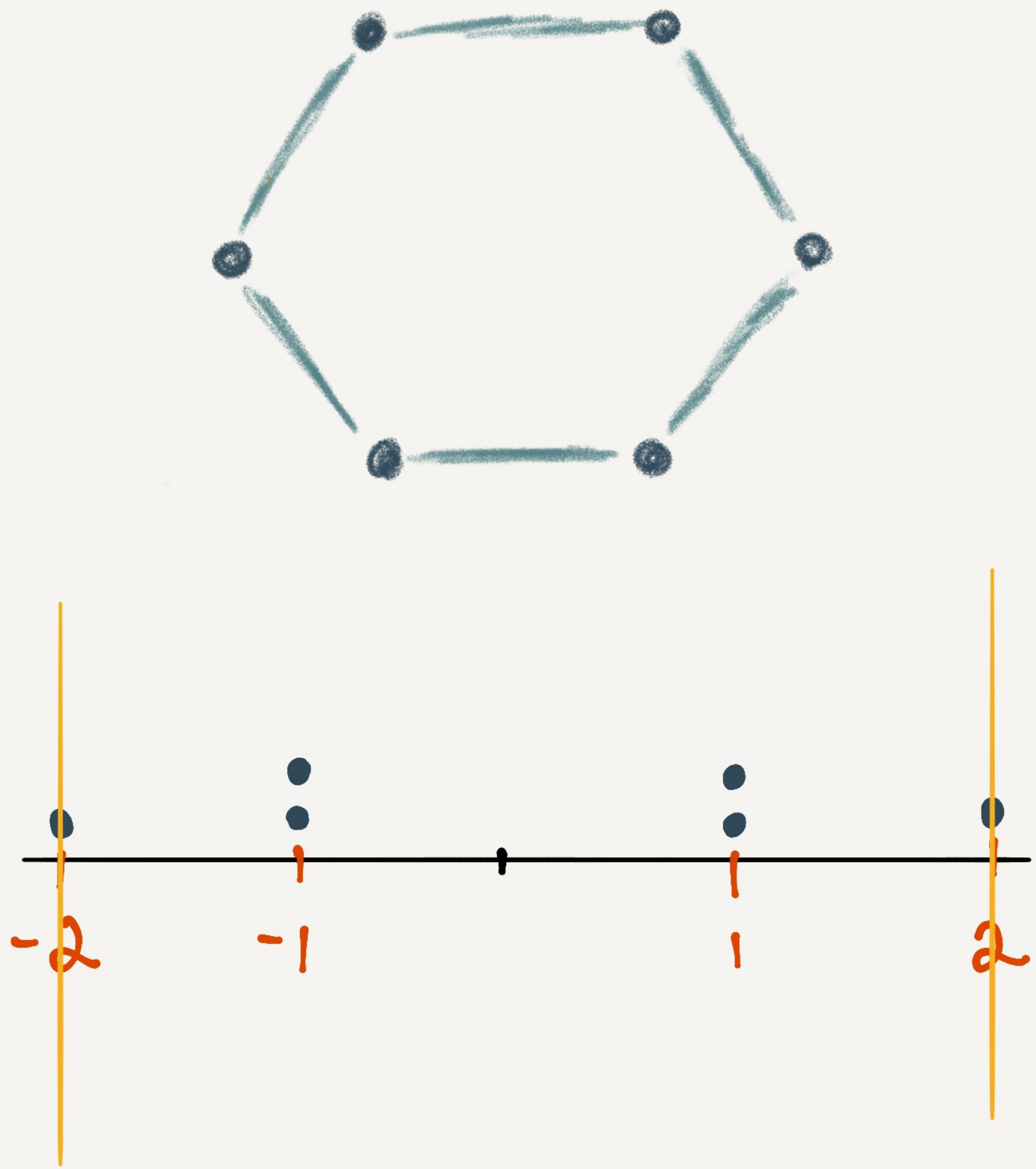
$\forall \varepsilon > 0$ only finitely many d-regular graphs G satisfy

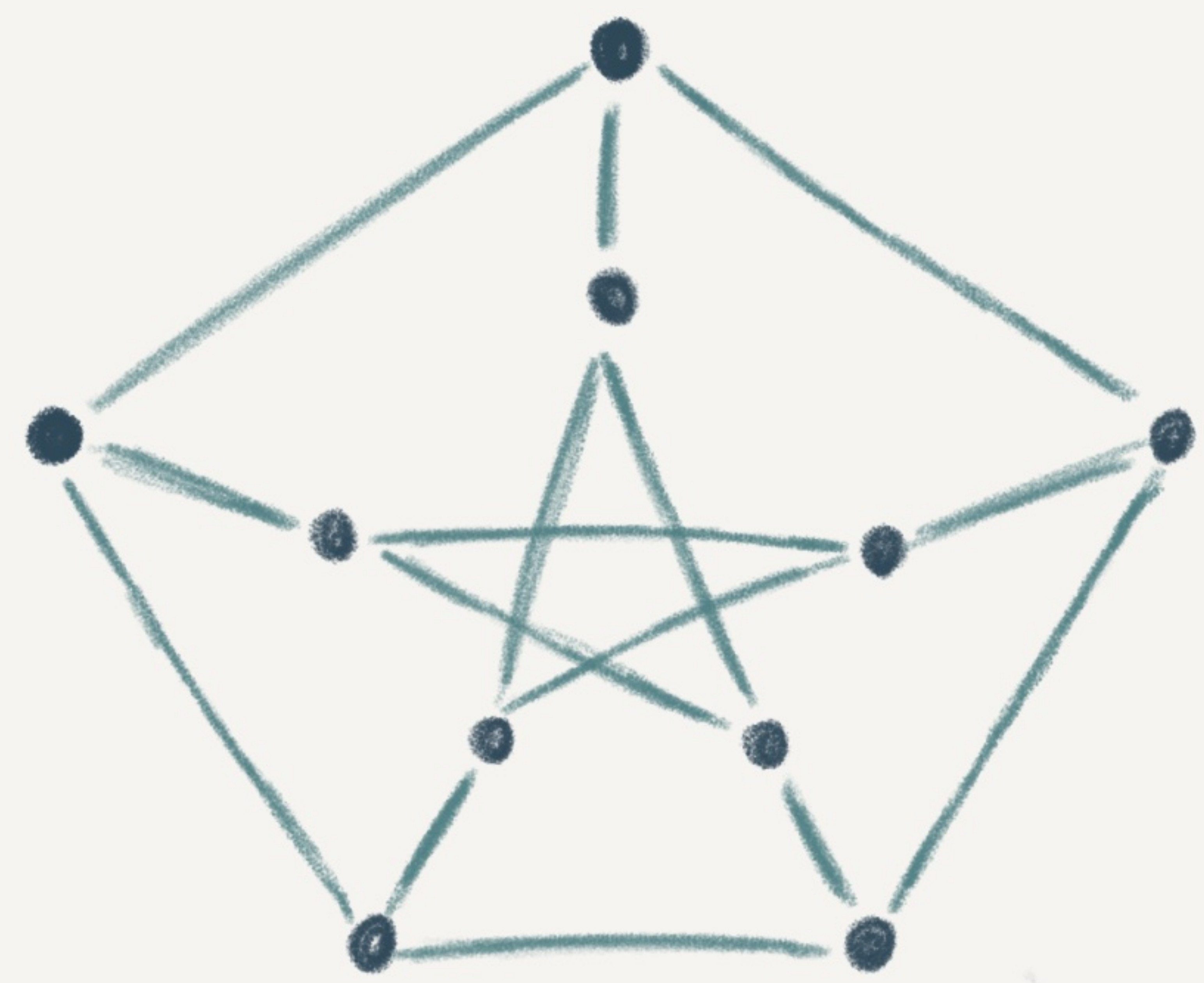
$$\lambda_G < 2\sqrt{d-1} - \varepsilon$$

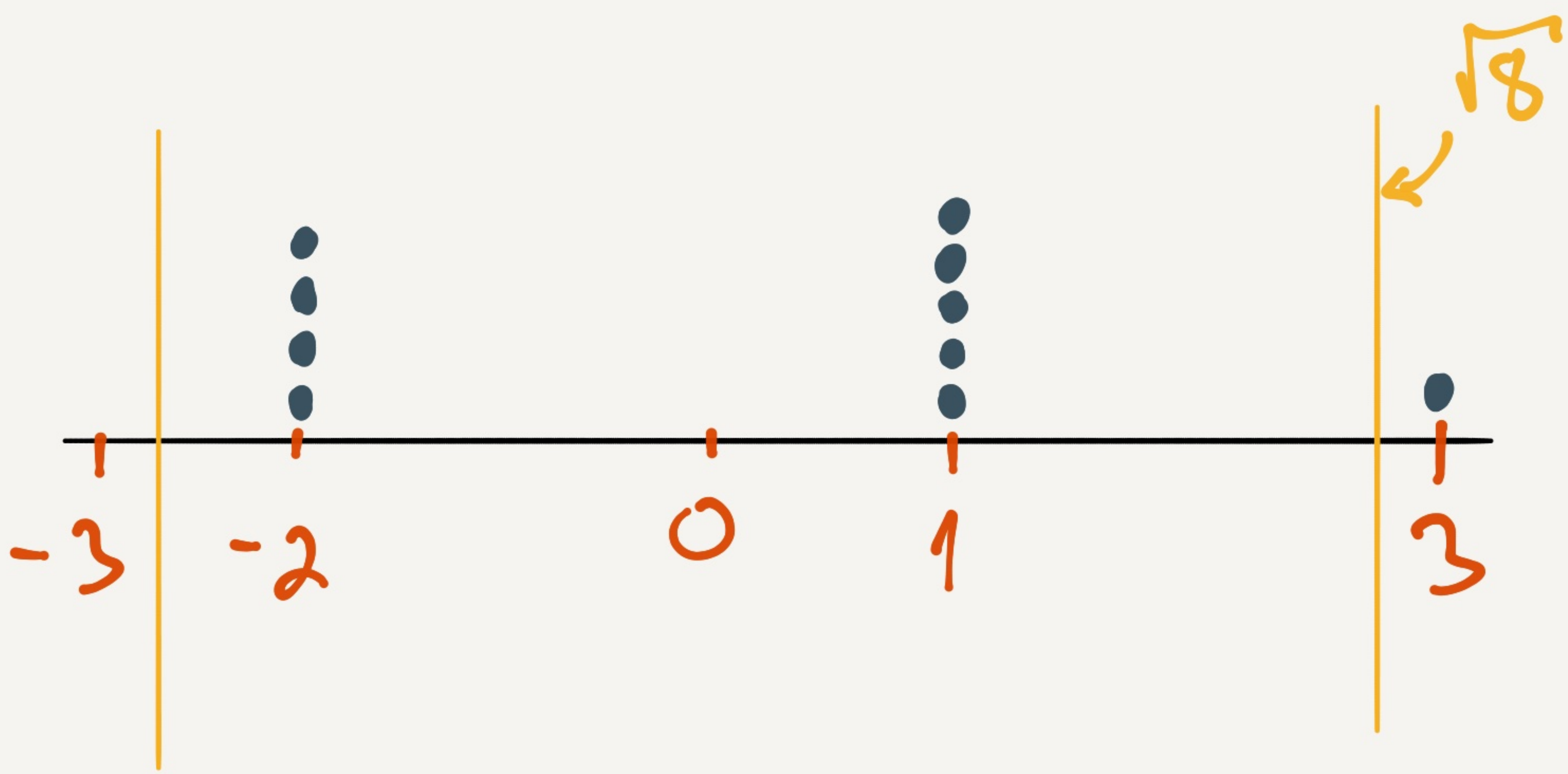
Def.

A d-regular graph G is Ramanujan if

$$\lambda_G \leq 2\sqrt{d-1}.$$


-2
-1
1
2





$\sqrt{8}$
-3
-2
0
1
3

A typical d-regular graph is known to be nearly Ramanujan $\left(2\sqrt{d-1}+o(1)\right)$. [F'08, ...]

How do we construct good expanders?

Number/group theory

[LPS'88, Mar'88]

Ramanujan!

Combinatorics & linear algebra

ZigZag [RVW'02...]

Lifting [BL'06...]

⋮

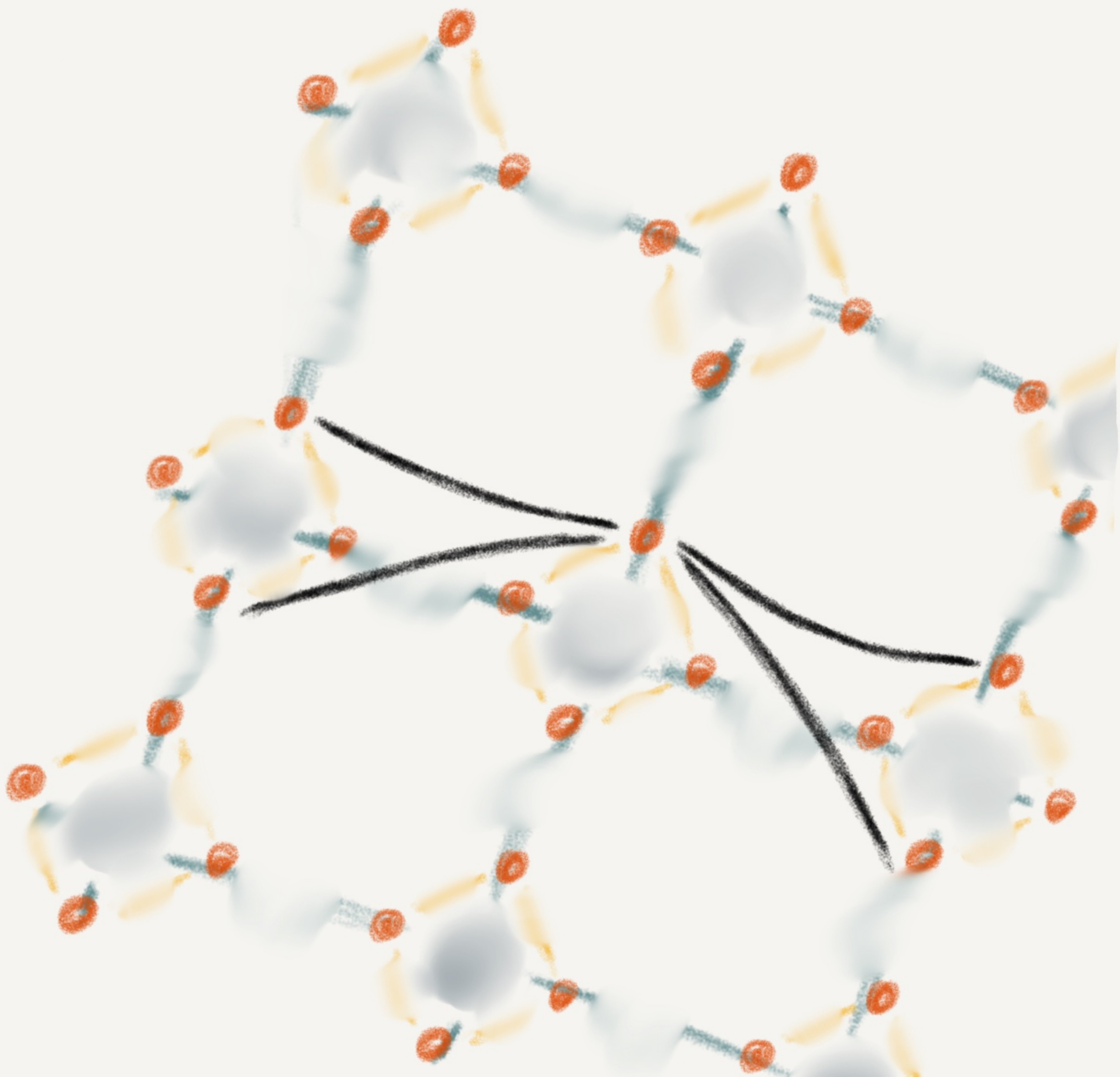
Analytic machinery (~free probability)

$\forall d$ [MSS'15...]

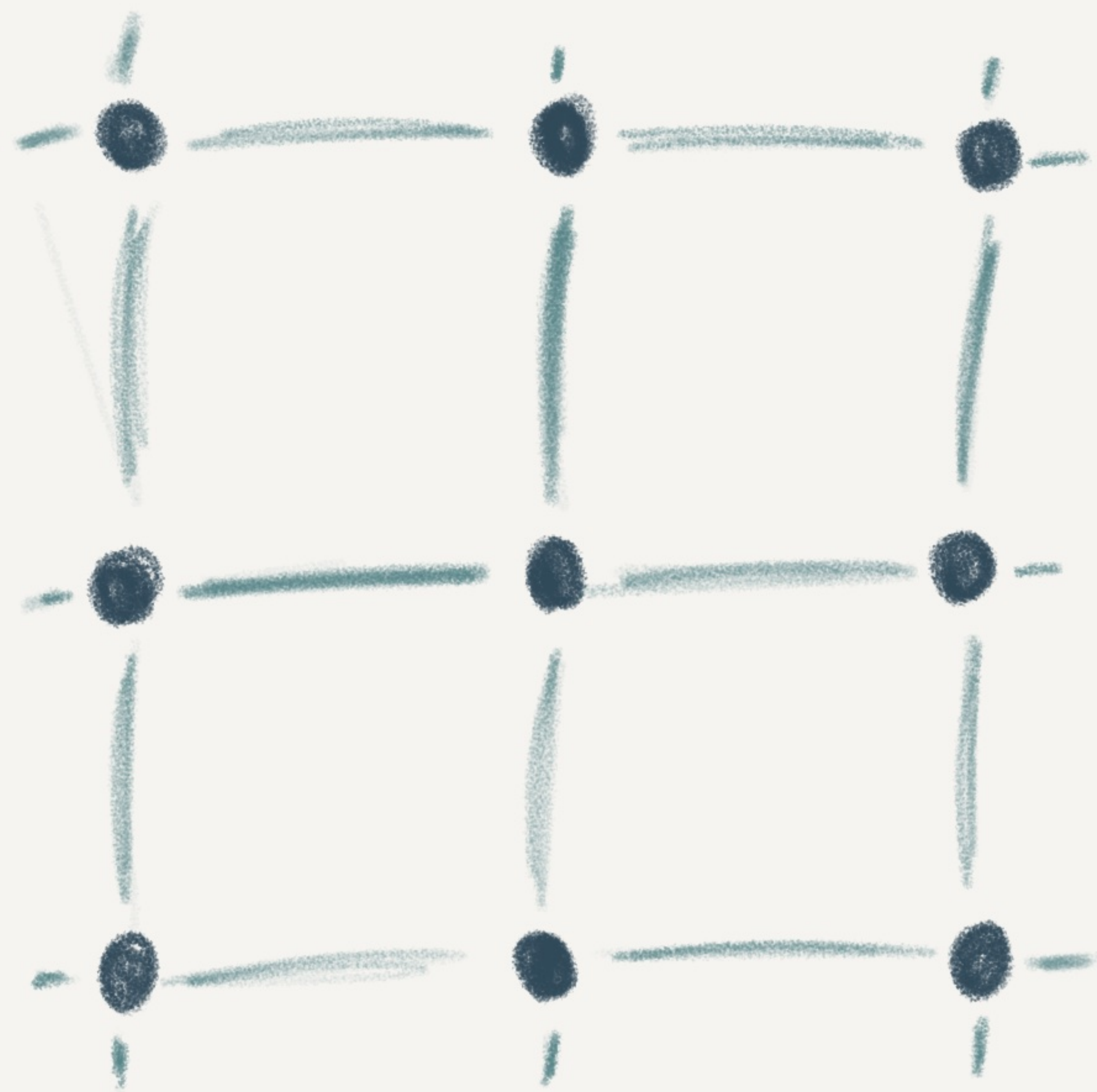
One-sided / Bipartite Ramanujan: $\lambda_2 \le 2\sqrt{d-1}$

The Zig Zag Product

[Reingold-Vadhan-Wigderson]



G

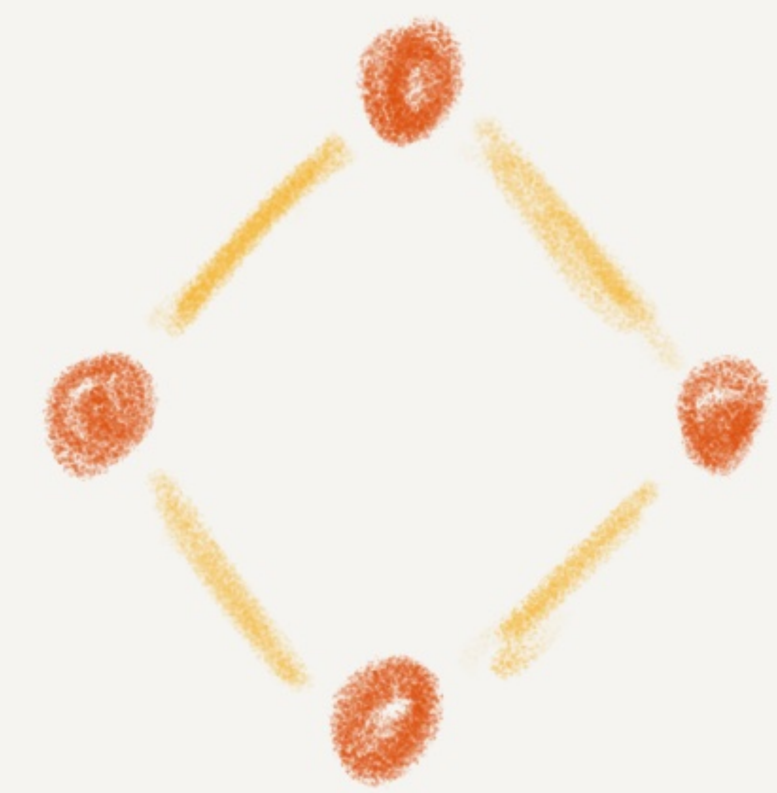


n vertices

d regular

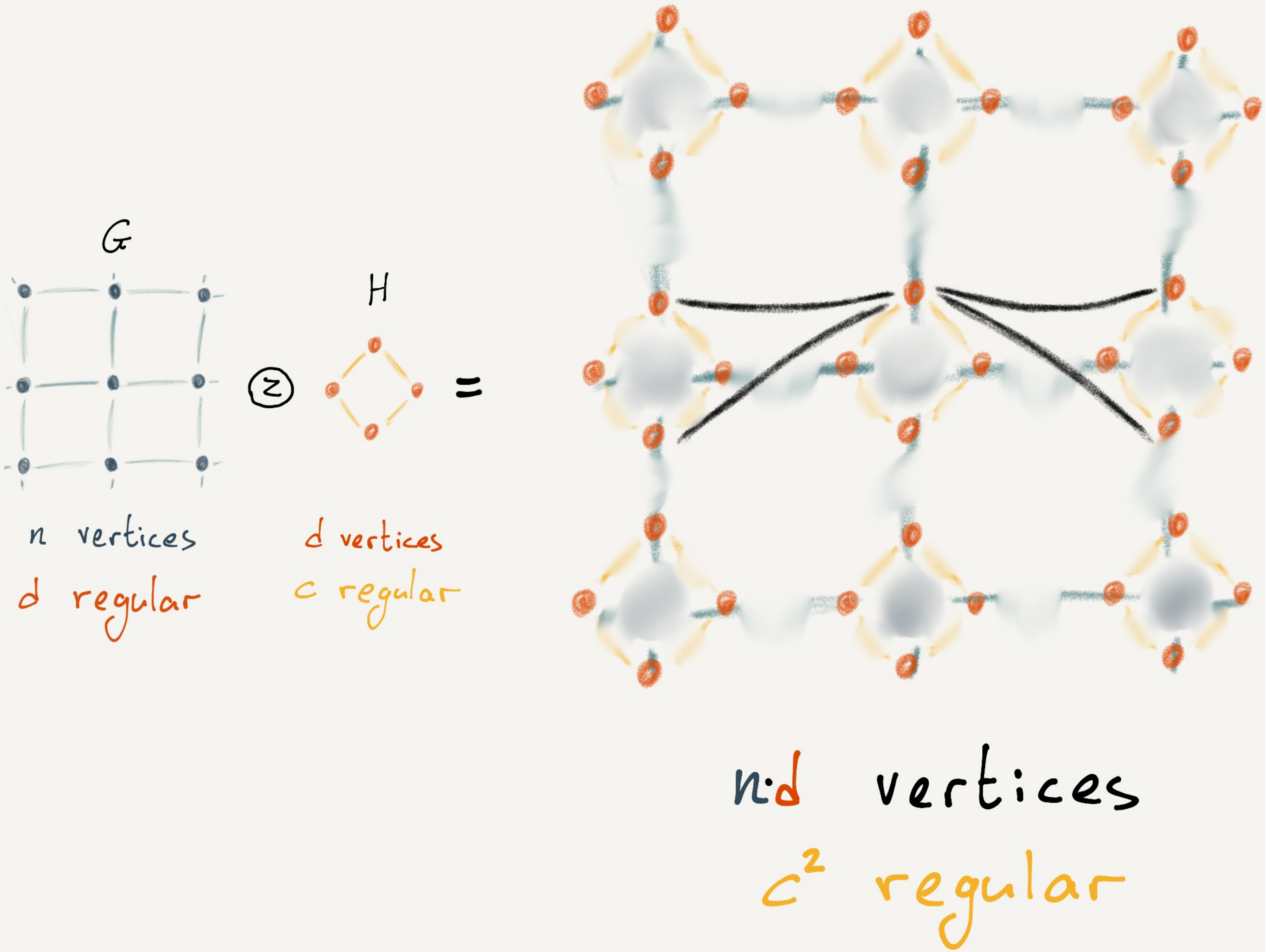
ⓩ

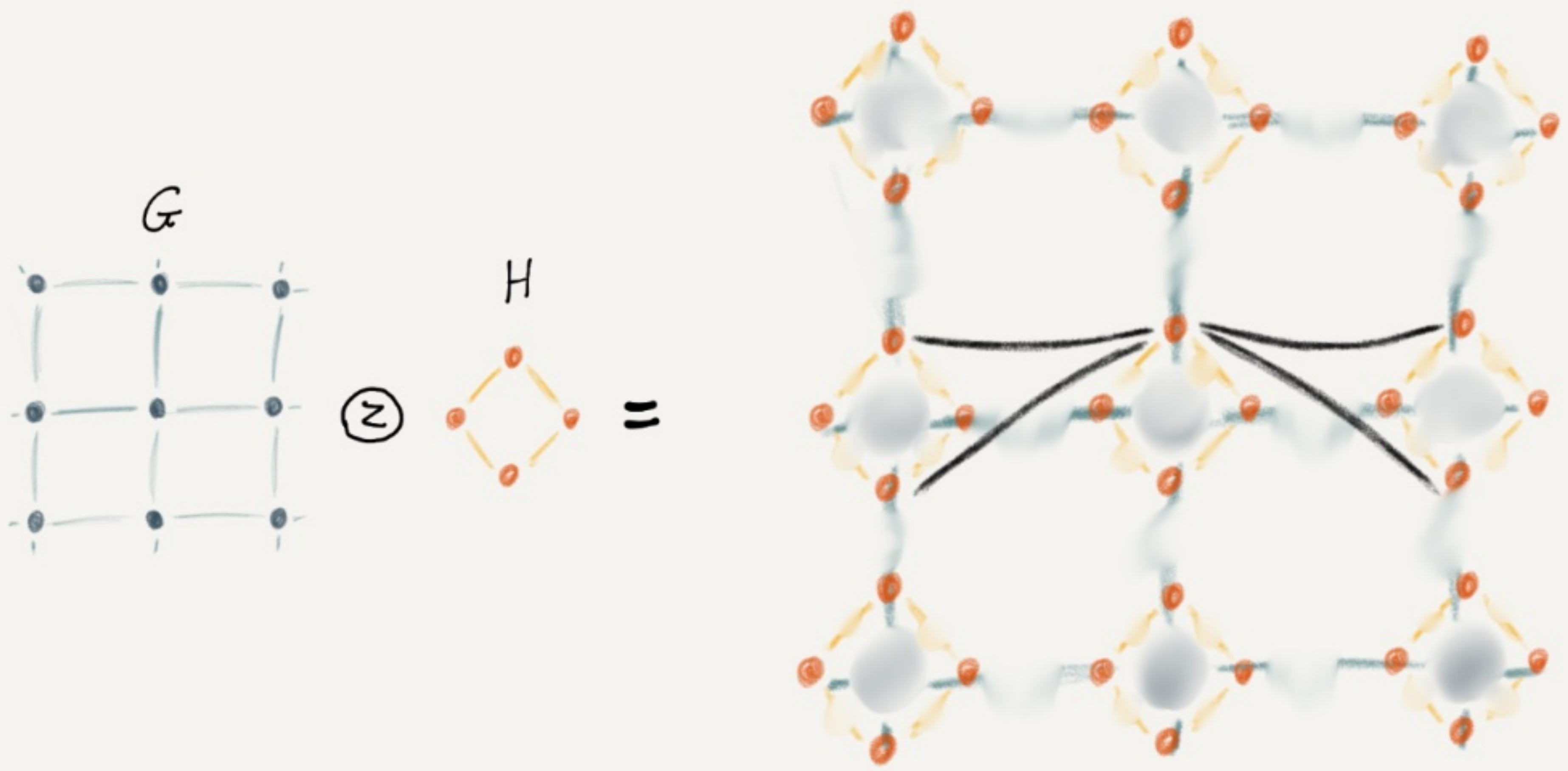
H



d vertices

c regular


G
H
n vertices
d regular
d vertices
c regular
n·d vertices
c² regular



$\omega \leftarrow \frac{\lambda}{\deg} \in [0,1]$

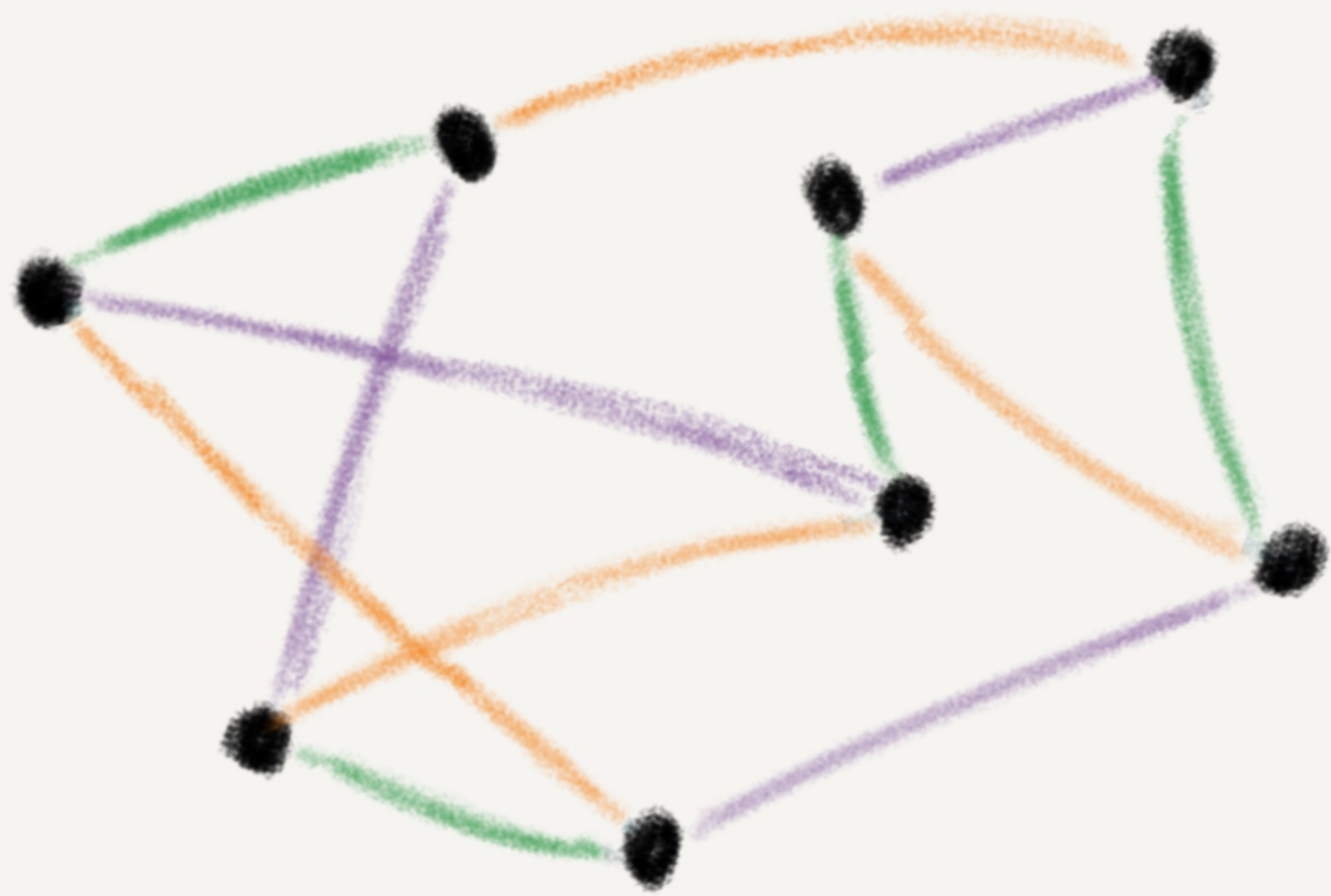
	$\lvert V \rvert$	degree	ω
G	n	d	ω_G
H	d	c	ω_H
$G ⓩ H$	$n \cdot d$	c^2	?

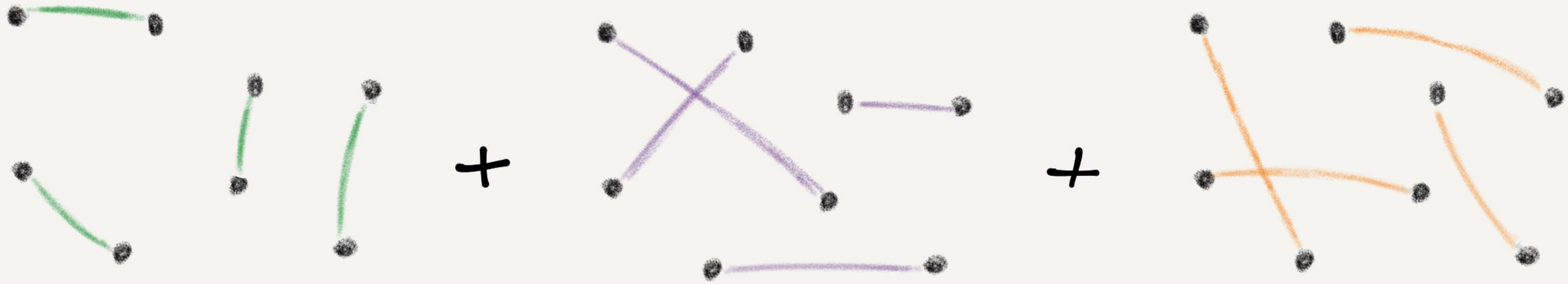
Thm [RVW].

$$\omega_{G ⓩ H} \lesssim \omega_G + \omega_H$$

Union of "free" perfect matchings

[Marcus-Spielman-Srivastava]





some fixed perfect matching

$$A_G := P_1^T M P_1 + P_2^T M P_2 + P_3^T M P_3$$

Permutation matrices

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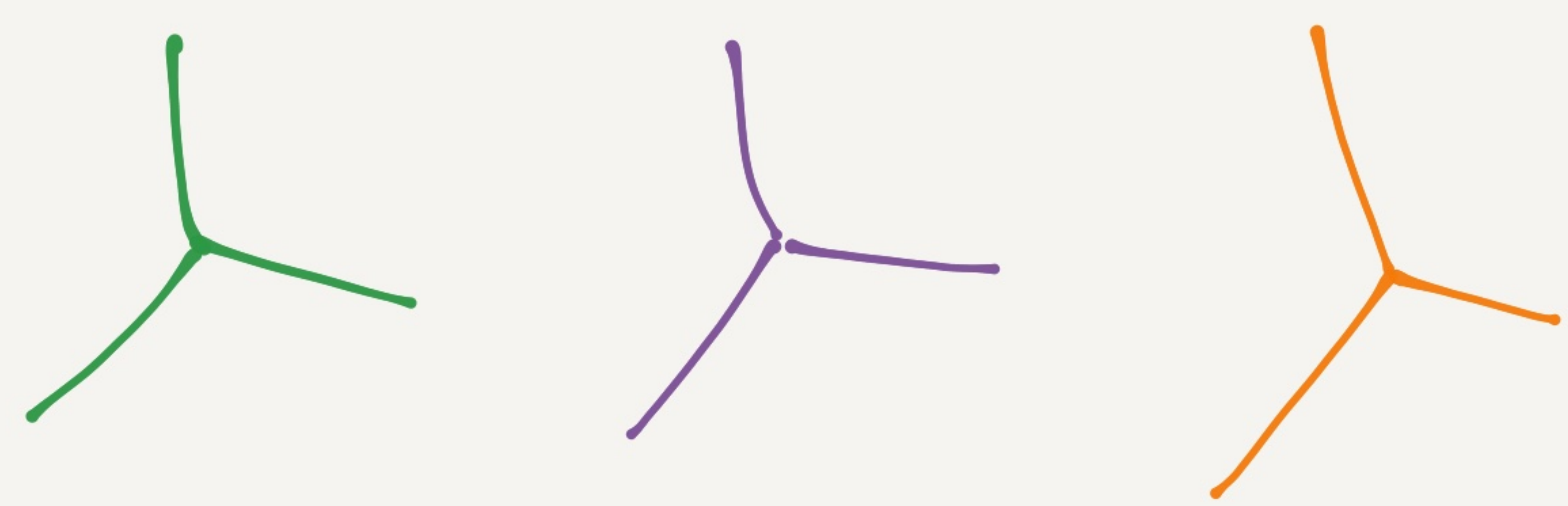
Permutation matrices

Question.

How should we choose P_1, P_2 & P_3 so as to minimize $\lambda(A_G)$?

$\mathrm{Spec}(P^T M P) = \pm 1$ so the entire game lies in the eigenvectors. We seem to want to decorrelate



"Random" enough so to "decorrelate" the eigenvectors of G, H.

$$G + Q^T H Q$$

Very random: Haar measure complete decorrelation but we don't end up with a graph

Somewhat random: random permutation. Is it random enough? Yields a graph!

MSS: yes!

Let $\chi_x(G)$ be G-s characteristic polynomial

$$\chi_x(G) = \prod x - \lambda_i$$

Theorem [MSS's Quadrature]

essentially...

$$\mathop{\mathbb{E}}_{Q} \chi_x(G + Q^T H Q) = \mathop{\mathbb{E}}_{P} \chi_x(G + P^T H P)$$

Beautiful... but is there a good P ?

Yes! MSS-s interlacing

Recap.

$$\mathop{\mathbb{E}}_{Q} \chi_x(G + Q^T H Q) \overset{\text{Quadrature}}{=} \mathop{\mathbb{E}}_{P} \chi_x(G + P^T H P)$$

$$\Downarrow \quad \text{Interlacing}$$

$\exists$ a "good" P, namely

$$\operatorname{maxroot} \chi_x(G + P^T H P) \leq \operatorname{maxroot} \mathop{\mathbb{E}}_{Q} \chi_x(G + Q^T H Q)$$

The question remains: Can we deduce

$$\mathop{\mathbb{E}}_{Q} \chi_x(G + Q^T H Q)$$

from $\chi_x(G)$, $\chi_x(H)$?

Finite Free Probability

additive finite free convolution

$$\mathop{\mathbb{E}}_{Q} \chi_x(G + Q^T H Q) = \chi_x(G) \boxplus \chi_x(H)$$

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To delve further see my course FPT & Ramanujan Graphs, TAed by Gal Maor

gilcohen.org/2024-fpt