Algebraic Geometry for Theoretical Computer Science

Assignment 5

Lecturer: Gil Cohen *Hand in date: December 4, 2014*

Instructions: Please write your solutions in LaTeX / Word or exquisite handwriting. Submission can be done individually or in pairs.

In the following exercise you are asked to prove a correspondence between places, valuation rings and discrete valuations, as stated in class.

1. Let F/K be a function field and P a place of F/K.
   (a) Prove that the function $v_P \colon F \to \mathbb{Z} \cup \{\infty\}$, as defined in class, is well-defined (namely, independent of the choice of the local parameter), and that it is a discrete valuation ring of F/K.
   (b) Prove that
   $$\begin{aligned} \mathcal{O}_P &= \{z \in F \mid v_P(z) \geq 0\}, \\ \mathcal{O}_P^\times &= \{z \in F \mid v_P(z) = 0\}, \\ P &= \{z \in F \mid v_P(z) > 0\}. \end{aligned}$$
   (c) Prove that an element $z \in F$ is a local parameter for P if and only if $v_P(z) = 1$.
   (d) Conversely, suppose that v is a discrete valuation of F/K. Prove that the set $P = \{z \in F \mid v(z) > 0\}$ is a place of F/K, and $\mathcal{O}_P = \{z \in F \mid v(z) \geq 0\}$ is the corresponding valuation ring.

In the following exercise you will be asked to study the valuation rings, places and discrete valuations of the rational function field. In the coming lecture we will see that the places that appear in the exercise are all the places of the rational function field!

2. Let $F = K(x)$ be the rational function field. Given an irreducible polynomial $p(x) \in K[x]$, consider the set
   $$\mathcal{O}_{p(x)} = \left\{ \frac{f(x)}{g(x)} \;\middle|\; f(x), g(x) \in K[x] \text{ such that } p(x) \nmid g(x) \right\}.$$
   (a) Prove that $\mathcal{O}_{p(x)}$ is a valuation ring of $K(x)/K$.
   (b) What is the corresponding maximal ideal $P_{p(x)}$?
   (c) What is the associated discrete valuation $v_{p(x)}$?

(d) Find a local parameter for $P_{p(x)}$.

Let

$$\mathcal{O}_\infty = \left\{ \frac{f(x)}{g(x)} \;\middle|\; f(x), g(x) \in K[x] \text{ such that } \deg f(x) \le \deg g(x) \right\}.$$

(e) Prove that $\mathcal{O}_\infty$ is a valuation ring of $K(x)/K$.

(f) What is the corresponding maximal ideal P_∞ ? P_∞ is called the infinite place of $K(x)$.

(g) What is the associated discrete valuation v_∞ ?

(h) Find a local parameter for P_∞.