Normal Function Field Extensions

Unit 19

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Overview

Group actions

Groups allow us to study the symmetries of an object though this connection is sometimes missed in the abstract study of groups. We quickly recap it.

Definition 1

Let X be a set and G a group. A group action α of G on X is a function

$$\alpha : G \times X \to X$$
$$(g, x) \mapsto gx$$

satisfying:

1. $ex = x$; and
2. $g(hx) = (gh)x$,

for all $x \in X$, $g, h \in G$.

The group G is said to act on X.

Group actions

Note that for every fixed $g \in G$, the function

$$\begin{aligned} \varphi_g : X &\to X \\ x &\mapsto gx \end{aligned}$$

is a bijection. Indeed,

$$\begin{aligned} \varphi_g(x) = \varphi_g(y) \quad &\Longrightarrow \quad gx = gy \\ &\Longrightarrow \quad g^{-1}(gx) = g^{-1}(gy) \\ &\Longrightarrow \quad (g^{-1}g)x = (g^{-1}g)y \\ &\Longrightarrow \quad ex = ey \\ &\Longrightarrow \quad x = y. \end{aligned}$$

Moreover,

$$\varphi_g(g^{-1}x) = g(g^{-1}x) = (gg^{-1})(x) = ex = x.$$

Group actions and symmetries

Informally, a symmetry of an object is doing something to it that does not change it.

Formally, A symmetry of a set X is a bijection $\varphi : X \to X$. The set of all symmetries of X, denoted $\mathrm{Sym}(X)$, is a group under composition.

By the previous observation, if G acts on X then every $g \in G$ gives rise to an element $\varphi_g \in \mathrm{Sym}(X)$. Moreover, the map

$$G \to \mathrm{Sym}(X)$$
$$g \mapsto \varphi_g$$

is a group homomorphism since $e \mapsto \mathrm{id}_X$ and

$$\varphi_{gh} = \varphi_g \varphi_h.$$

Indeed, for every $x \in X$,

$$\varphi_{gh}(x) = (gh)x = g(hx) = g\varphi_h(x) = \varphi_g(\varphi_h(x)) = (\varphi_g\varphi_h)(x).$$

From here on we denote φ_g by g.

Transitive actions and orbits

Definition 2

An action of G on $X \neq \emptyset$ is called transitive if

$$\forall x, y \in X \quad \exists g \in G \quad \text{s.t.} \quad gx = y.$$

Definition 3

The orbit of an element $x \in X$ under the action of G is

$$Gx = \{gx \mid g \in G\}.$$

The orbits form a partition of X, hence, they give rise to an equivalence relation:

$$x \sim y \quad \iff \quad Gx = Gy.$$

Note that a group action is transitive iff it has a single orbit.

Definition 4

The set of all orbits of X under the action of G, denoted X/G, is called the quotient of the action.

The stabilizer

When $gx = x$ we say that g fixes x or that x is a fixed point of g.

Definition 5

For $x \in X$ the stabilizer subgroup of G with respect to x is given by

$$G_x = \{g \in G \mid gx = x\}.$$

Observe that for $x, y \in X$ s.t. $y = gx$ the two stabilizers satisfy

$$G_y = gG_xg^{-1}.$$

Thus, the stabilizers of elements in the same orbit are conjugate to each other.

The orbit-stabilizer theorem

Fix $x \in X$. Then,

$$\begin{aligned} gx = hx \quad &\iff \quad h^{-1}gx = x \\ &\iff \quad h^{-1}g \in G_x \\ &\iff \quad gG_x = hG_x. \end{aligned}$$

So

$$|Gx| = |(G : G_x)|.$$

This is despite the fact that G_x may not be normal (so G/G_x is not a group). For finite groups,

$$|x\text{'s orbit}| = |Gx| = \frac{|G|}{|G_x|} = \frac{|G|}{|x\text{'s stabilizer}|}.$$

This result is called the orbit-stabilizer theorem.

Overview

Normal field extensions

Definition 6 (Normal field extensions)

An algebraic extension L/K is said to be normal if every irreducible polynomial $f(x) \in \mathsf{K}[x]$ that has a root $\alpha \in \mathsf{L}$ factors to linear factors in $\mathsf{L}[x]$.

Theorem 7

Let L/K be an algebraic extension and assume $\mathsf{L} \subseteq \bar{\mathsf{K}}$. TFAE

1. L/K *is normal.*
2. L *is the splitting field of some* $\{f_\alpha(x) \in \mathsf{K}[x]\}_\alpha$.
3. *Every automorphism of* $\bar{\mathsf{K}}/\mathsf{K}$ *maps* L *to* L.
4. *For every* $\alpha \in \mathsf{L}$*, all conjugates of* α *over* K *are in* K.

Overview

Separable polynomials

Definition 8 (Separability without referring to an algebraic closure)

Let K be a field. An irreducible polynomial $f(x) \in \mathrm{K}[x]$ is separable if it is has distinct roots in any field extension of K.

Definition 9 (Separability by referring to an algebraic closure)

Let K be a field and $\bar{\mathrm{K}}$ an algebraic closure of K. An irreducible polynomial $f(x) \in \mathrm{K}[x]$ is separable if it is a product of distinct linear factors in $\bar{\mathrm{K}}[x]$.

In characteristic zero all irreducible polynomials are separable. So from here on, we denote the characteristic by $p > 0$.

Separable elements

Definition 10

Let F/E be a field extension. An element $\alpha \in$ F is separable over E if α is algebraic over E and its minimal polynomial is separable over E.

It is known that

$$\alpha, \beta \text{ are separable} \quad \Longrightarrow \quad \alpha+\beta,\ \alpha\beta,\ \alpha^{-1} \text{ are separable.}$$

Thus, the set of elements in F that are separable over E form a field, denoted by E_s.

Separable and purely inseparable extensions

If $\alpha \in \mathrm{F}$ it not separable then $\exists e \geq 1$ s.t. $\alpha^{p^e} \in \mathrm{E}_s$. Further, the minimal polynomial of α over E_s is $(x-\alpha)^{p^e}$.

The extension F/E_s is purely inseparable, namely, every $\alpha \in \mathrm{F} \setminus \mathrm{E}_s$ is not separable over E_s.

Every algebraic field extension F/E can be decomposed as

$$\begin{tikzcd}
F \arrow[d, dash, "\text{purely inseparable}"] \\
E_s \arrow[d, dash, "\text{max sep extension}"] \\
E
\end{tikzcd}$$

We denote $q = [\mathrm{F} : \mathrm{E}]_i = [\mathrm{F} : \mathrm{E}_s]$.

Overview

Galois extensions

Let F/E be an algebraic field extension. Denote by $\Gamma = \mathrm{Aut}(\mathsf{F}/\mathsf{E})$ the set of automorphisms of F that fix each element of E.

Define the field

$$\mathsf{F}^{\Gamma} = \{x \in \mathsf{F} \mid \forall \sigma \in \Gamma \ \ \sigma(x) = x\}.$$

By Galois Theory,

$$\mathsf{F}^{\Gamma} = \mathsf{E} \quad \Longleftrightarrow \quad \mathsf{F}/\mathsf{E} \text{ is normal and separable.}$$

In such case, F/E is called a Galois extension. For such extensions, the group Γ is denoted $\mathrm{Gal}(\mathsf{F}/\mathsf{E})$. It holds that

$$|\mathrm{Gal}(\mathsf{F}/\mathsf{E})| = [\mathsf{F} : \mathsf{E}].$$

Separable and purely inseparable extensions

Assume F/E is normal and consider E_s as before. We have that q is some power of the characteristic, and that $F^q = E_s$.

$$
\begin{tikzcd}
F \arrow[d, dash, "\text{purely inseparable}"] \\
E_s \arrow[d, dash, "\text{max sep extension}"] \\
E
\end{tikzcd}
$$

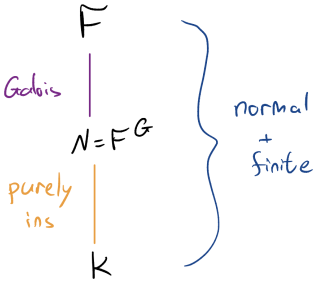
Further, E_s/E is a Galois extension and Aut(F/E) can be identified with the Galois group $G = \mathrm{Gal}(E_s/E)$.

A useful lemma

Lemma 11

Let F/K *be a finite normal extension. Let* $G = \mathrm{Aut}(\mathrm{F}/\mathrm{K})$ *and denote* $\mathrm{N} = \mathrm{F}^G$. *Then,*

1. F/N *is Galois; and*
2. N/K *is purely inseparable.*
3. $\mathrm{Gal}(\mathrm{F}/\mathrm{N}) = G$.



A useful lemma

Proof.

Starting with Item 1, clearly, F/N is normal, and so we focus on separability.

Take $\alpha \in \mathrm{F}$ and let $f(x) \in \mathrm{N}(x)$ be its minimal polynomial over N. Let

$$\alpha = \alpha_1, \alpha_2, \ldots, \alpha_m$$

be the distinct roots of $f(x)$. By normality of F/N these lie in F. Define

$$g(x) = \prod_{i=1}^{m} (x - \alpha_i) \in \mathrm{F}[x].$$

Fix $\sigma \in G$ and $i \in [m]$. Then,

$$0 = \sigma(0) = \sigma(f(\alpha_i)) = f(\sigma(\alpha_i)),$$

where the last equality holds since $\mathrm{N} = \mathrm{F}^G$. Thus, $\sigma(\alpha_i) = \alpha_j$. But σ is one to one and so it acts as a bijection on $\alpha_1, \ldots, \alpha_m$.

A useful lemma

Proof.

Therefore,

$$\forall \sigma \in G \quad \sigma g(x) = \prod_{i=1}^{m} (x - \sigma\alpha_i) = \prod_{i=1}^{m} (x - \alpha_i) = g(x).$$

Hence,

$$g(x) \in \mathrm{F}^G[x] = \mathrm{N}[x].$$

But $f(x)$ is the minimal polynomial of α over N and so $f(x) \mid g(x)$. Clearly, however, $g(x) \mid f(x)$ and so $f(x) = g(x)$ is separable. Thus, α is separable over N.

A useful lemma

Proof.

Moving on to Item 2, we know that

$$[\mathsf{F} : \mathsf{K}]_s = [\mathsf{F} : \mathsf{N}]_s[\mathsf{N} : \mathsf{K}]_s.$$

But by Item 1,

$$[\mathsf{F} : \mathsf{N}]_s = [\mathsf{F} : \mathsf{N}].$$

Galois Theory tells us that

$$[\mathsf{F} : \mathsf{N}] = |\mathrm{Gal}(\mathsf{F}/\mathsf{N})| = |\mathrm{Gal}(\mathsf{F}/\mathsf{F}^G)| = |G| = |\mathrm{Aut}(\mathsf{F}/\mathsf{K})| = [\mathsf{F} : \mathsf{K}]_s.$$

Thus,

$$[\mathsf{F} : \mathsf{N}] = [\mathsf{F} : \mathsf{K}]_s = [\mathsf{F} : \mathsf{N}]_s[\mathsf{N} : \mathsf{K}]_s = [\mathsf{F} : \mathsf{N}][\mathsf{N} : \mathsf{K}]_s.$$

Therefore, $[\mathsf{N} : \mathsf{K}]_s = 1$, namely, N/K is purely inseparable.

Item 3 follows by the above. □

Overview

Isomorphism between function fields

Let F/L, F'/L' be function fields. Let $\sigma : \mathsf{F} \to \mathsf{F}'$ be an isomorphism s.t. $\sigma(\mathsf{L}) = \mathsf{L}'$.

For every valuation υ of F there is a corresponding valuation of F', denoted by $\sigma\upsilon$, that is defined as follows: for $x \in \mathsf{F}'$,

$$(\sigma\upsilon)(x) = \upsilon(\sigma^{-1}(x)).$$

Verify that this is indeed a valuation, and note that

$$\begin{aligned}\mathcal{O}_{\sigma\upsilon} &= \{x \in \mathsf{F}' \mid \sigma\upsilon(x) \geq 0\} \\ &= \{x \in \mathsf{F}' \mid \upsilon(\sigma^{-1}(x)) \geq 0\} \\ &= \{\sigma(y) \mid y \in \mathsf{F} \text{ and } \upsilon(y) \geq 0\} \\ &= \sigma(\mathcal{O}_\upsilon).\end{aligned}$$

Similarly, $\mathfrak{m}_{\sigma\upsilon} = \sigma(\mathfrak{m}_\upsilon)$.

Isomorphism between function fields

By the above, equivalent valuations are mapped by σ to equivalent valuations.

Further, a valuation that is trivial on L is mapped by σ to a valuation that is trivial on L' and vice versa.

Recall that a prime divisor $\mathfrak{p}$ of F/L is an equivalence class of places of F that are trivial on L. Thus, by the above, σ induces a bijection

$$\mathfrak{p} \mapsto \sigma\mathfrak{p}$$

between the prime divisors of F/L and the prime divisors of F'/L'.

Isomorphism between function fields

As the diagram below depicts, σ also induces an isomorphism $\bar{\sigma}$ between the residue fields

$$\bar{\sigma} : \mathsf{F}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}} \big/ \mathfrak{m}_{\mathfrak{p}} \to \mathsf{F}_{\sigma\mathfrak{p}} = \mathcal{O}_{\sigma\mathfrak{p}} \big/ \mathfrak{m}_{\sigma\mathfrak{p}}$$

that is given by $\bar{\sigma}\bar{x} = \overline{\sigma x}$ or, more informatively, $\bar{\sigma}(x + \mathfrak{m}_{\mathfrak{p}}) = \sigma x + \mathfrak{m}_{\sigma\mathfrak{p}}$.

$$\begin{tikzcd}
\mathcal{O}_{\mathfrak{p}} \arrow[r, "\sigma"] \arrow[d] & \mathcal{O}_{\sigma\mathfrak{p}} \arrow[d] \\
\mathcal{O}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} = \mathsf{F}_{\mathfrak{p}} \arrow[r, "\bar{\sigma}"] & \mathsf{F}_{\sigma\mathfrak{p}} = \mathcal{O}_{\sigma\mathfrak{p}}/\mathfrak{m}_{\sigma\mathfrak{p}}
\end{tikzcd}$$

In particular, $\deg(\sigma\mathfrak{p}) = \deg(\mathfrak{p})$.

Overview

Normal extensions of function fields

Definition 12

A function field extension F/L over E/K is called normal if F/E is a normal field extension.

Claim 13

If F/L is a normal extension of E/K then L/K is normal.

To prove Claim 13 we first prove

Claim 14

Let E/K be a field extension s.t. K is algebraically closed in E. Then,

$$f \in K[x] \text{ is irreducible} \quad \Longrightarrow \quad f \text{ is irreducible in } E[x].$$

Normal extensions of function fields

Proof.

Let $g(x) \in \mathrm{E}[x]$ be an irreducible factor of $f(x)$ in $\mathrm{E}[x]$. We will prove that $g(x) \in \mathrm{K}[x]$.

In $\bar{\mathrm{E}}[x]$ we can write

$$g(x) = \prod_{i=1}^{m} (x - a_i).$$

But the a_i-s are some of the roots of $f(x)$, and so $a_i \in \bar{\mathrm{K}}$.

The coefficients of $g(x)$ are polynomials in the a_i-s, and so

$$g(x) \in \bar{\mathrm{K}}[x].$$

But $g(x) \in \mathrm{E}[x]$, and so

$$g(x) \in (\bar{\mathrm{K}} \cap \mathrm{E})[x] = \mathrm{K}[x].$$

$\square$

Proof of Claim 13.

We already proved that

$$\mathsf{F}/\mathsf{E} \text{ is algebraic} \quad \Longrightarrow \quad \mathsf{L}/\mathsf{K} \text{ is algebraic.}$$

Take $f(x) \in \mathsf{K}[x]$ irreducible with a root $\alpha \in \mathsf{L}$. We need to prove that $f(x)$ splits in $\mathsf{L}[x]$.

By Claim 14, $f(x)$ is irreducible over E. Further, $\alpha \in \mathsf{L} \subseteq \mathsf{F}$, and so as F/E is normal, $f(x)$ splits over F.

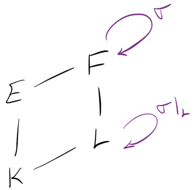
Now, $f(x) \in \mathsf{K}[x] \subseteq \mathsf{L}[x]$ and so all roots of $f(x)$ in F are algebraic over L, and so they belong to L. $\square$

$$\begin{tikzcd}
 & F \arrow[d, dash] \\
E \arrow[ur, dash] \arrow[d, dash] & L \\
K \arrow[ur, dash] &
\end{tikzcd}$$

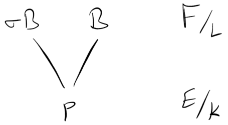
Normal extensions of function fields

Assume again that F/L is a normal extension of E/K.

Consider $\sigma \in \mathrm{Aut}(\mathsf{F}/\mathsf{E})$. In particular, $\sigma|_K = \mathrm{id}_\mathsf{K}$. As L/K is normal, we have that $\sigma(\mathsf{L}) = \mathsf{L}$. Namely, $\sigma|_\mathsf{L} \in \mathrm{Aut}(\mathsf{L}/\mathsf{K})$.



As $\sigma|_\mathsf{E} = \mathrm{id}_\mathsf{E}$ we have that for a prime divisor $\mathfrak{P}$ of F/L lying over a prime divisor $\mathfrak{p}$ of E/K it holds that $\sigma\mathfrak{P}$ is also a prime divisor of F/L that lies over $\sigma\mathfrak{p} = \mathfrak{p}$.



Normal extensions of function fields

$$
\begin{tikzcd}
L \arrow[r, dash] \arrow[d, dash] & F_{\mathfrak{P}} \arrow[d, dash] \\
K \arrow[r, dash] & E_{\mathfrak{p}}
\end{tikzcd}
$$

We have that

$$
\begin{aligned}
f(\sigma\mathfrak{P}/\mathfrak{p}) &= [\mathsf{L} : \mathsf{K}] \cdot \frac{\deg \sigma\mathfrak{P}}{\deg \mathfrak{p}} = [\mathsf{L} : \mathsf{K}] \cdot \frac{\deg \mathfrak{P}}{\deg \mathfrak{p}} = f(\mathfrak{P}/\mathfrak{p}), \\
e(\sigma\mathfrak{P}/\mathfrak{p}) &= (\upsilon_{\sigma\mathfrak{P}}(\mathsf{F}^{\times}) : \upsilon_{\sigma\mathfrak{p}}(\mathsf{E}^{\times})) = (\upsilon_{\mathfrak{P}}(\mathsf{F}^{\times}) : \upsilon_{\mathfrak{p}}(\mathsf{E}^{\times})) = e(\mathfrak{P}/\mathfrak{p}).
\end{aligned}
$$

Thus, for every prime divisor $\mathfrak{p}$ of E/K, $\mathrm{Aut}(\mathsf{F}/\mathsf{E})$ acts on the prime divisors lying over $\mathfrak{p}$, keeping the residual degree and ramification index intact.

We turn to prove that this action is transitive.

Normal extensions of function fields

From hereon, let F/L be a normal finite extension of E/K. We let E_s be the maximal separable extension of E in F and denote

$$q = [\mathrm{F} : \mathrm{E}]_i = [\mathrm{F} : \mathrm{E}_s].$$

From Galois Theory, we know that q is some power of the characteristic p, and that $\mathrm{F}^q = \mathrm{E}_s$.

Further, E_s/E is a Galois extension and $\mathrm{Aut}(\mathrm{F}/\mathrm{E})$ can be identified with the Galois group $G = \mathrm{Gal}(\mathrm{E}_s/\mathrm{E})$.

$$\begin{tikzcd} F \arrow[d, dash, "\text{purely inseparable}"] \\ E_s \arrow[d, dash, "\text{max sep extension}"] \\ E \end{tikzcd}$$

Normal extensions of function fields

If $z \in \mathsf{E}_s$ then

$$\prod_{\sigma \in G} \sigma z$$

is fixed by all elements of $G = \mathrm{Gal}(\mathsf{E}_s/\mathsf{E})$ since for every $\tau \in G$,

$$\tau \prod_{\sigma \in G} \sigma z = \prod_{\sigma \in G} \tau\sigma z = \prod_{\sigma \in G} \sigma z.$$

Thus, $\prod_{\sigma \in G} \sigma z \in \mathsf{E}_s^G = \mathsf{E}$.

For every $x \in \mathsf{F}$ we have that $x^q \in \mathsf{E}_s$, and so

$$\prod_{\sigma \in G} \sigma(x^q) \in \mathsf{E}.$$

This way one can define the norm of finite normal extensions:

$$\mathrm{N}_{\mathsf{F}/\mathsf{E}}(x) \triangleq \left(\prod_{\sigma \in G} \sigma x\right)^q \in \mathsf{E}.$$

Normal extensions of function fields

Theorem 15

Let F/L *be a finite normal function field extension of* E/K*. Let* $\mathfrak{p}$ *be a prime divisor of* E/K*. Then,* Aut(F/E) *acts transitively on the set of prime divisors lying over* $\mathfrak{p}$*.*

That is, for every two prime divisors $\mathfrak{P}, \mathfrak{P}'$ *of* F/L *that lie over* $\mathfrak{p}$*,*

$$\exists \sigma \in \mathrm{Aut}(\mathrm{F}/\mathrm{E}) \quad s.t. \quad \sigma\mathfrak{P} = \mathfrak{P}'.$$

Proof.

Assume that $\mathfrak{P}' \neq \sigma\mathfrak{P}$ for all $\sigma \in G = \mathrm{Aut}(\mathrm{F}/\mathrm{E})$. Then, the corresponding orbits are disjoint

$$\{\sigma\mathfrak{P}' \mid \sigma \in G\} \cap \{\sigma\mathfrak{P} \mid \sigma \in G\} = \emptyset.$$

By the WAT $\exists x \in \mathrm{F}$ s.t.

$$\forall \sigma \in G \quad v_{\sigma\mathfrak{P}}(x) > 0 \;\text{ and }\; v_{\sigma\mathfrak{P}'}(x) < 0.$$

Normal extensions of function fields

Proof.

Recall that $q = [\mathsf{F} : \mathsf{E}_s]$, $G = \mathrm{Gal}(\mathsf{E}_s/\mathsf{E})$, and consider

$$y = \mathrm{N}_{\mathsf{F}/\mathsf{E}}(x) = \left(\prod_{\sigma \in G} \sigma x \right)^q \in \mathsf{E}.$$

Then,

$$\upsilon_{\mathfrak{P}}(y) = q \sum_{\sigma \in G} \upsilon_{\mathfrak{P}}(\sigma x) = q \sum_{\sigma \in G} \upsilon_{\sigma^{-1}\mathfrak{P}}(x) = q \sum_{\sigma \in G} \upsilon_{\sigma\mathfrak{P}}(x) > 0.$$

As $y \in \mathsf{E}$, we can also consider

$$\upsilon_{\mathfrak{p}}(y) = \frac{1}{e(\mathfrak{P}/\mathfrak{p})} \upsilon_{\mathfrak{P}}(y) > 0.$$

However, by considering $\upsilon_{\mathfrak{P}'}$ instead of $\mathfrak{P}$ we will reach the opposite conclusion, $\upsilon_{\mathfrak{p}}(y) < 0$, and the proof follows. □

Overview

The decomposition group

Again let $G = \mathrm{Aut(F/E)}$, $\mathfrak{p}$ as above and $\mathfrak{P}$ a prime divisor lying over $\mathfrak{p}$.

The stabilizer of $\mathfrak{P}$ is called the decomposition group

$$D(\mathfrak{P}) = \{\sigma \in G \mid \sigma\mathfrak{P} = \mathfrak{P}\}.$$

Since G acts transitively on the prime divisors lying over $\mathfrak{p}$,

$$r \triangleq |G\mathfrak{P}| = \text{number of prime divisors of F/L lying over } \mathfrak{p}.$$

Thus, by the orbit-stabilizer theorem,

$$r = (G : D(\mathfrak{P})). \tag{1}$$

Residue fields in normal extensions

Theorem 16

Let F/L *be an extension of* E/K*, and consider prime divisors* $\mathfrak{P}/\mathfrak{p}$*. Assume* F/E *is normal and finite. Then, the extension of the residue fields* $\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}}$ *is normal and finite.*

We already saw that $\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}}$ is finite since F/E is. In particular, $\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}}$ is algebraic. We turn to prove normality. To this end, we start by proving the following claim.

Claim 17

For every $z \in \mathsf{F}_{\mathfrak{P}} = \mathcal{O}_{\mathfrak{P}}/\mathfrak{m}_{\mathfrak{P}}$ there is a representative $y \in \mathcal{O}_{\mathfrak{P}}$ s.t.

1. $\upsilon_{\mathfrak{P}}(\sigma y) \geq 0$ for all $\sigma \in G$; and
2. $\upsilon_{\mathfrak{P}}(\sigma y) > 0$ for all $\sigma \in G \setminus D(\mathfrak{P})$.

Residue fields in normal extensions

Proof. (of Claim 17)

Take any representative $y' \in \mathcal{O}_{\mathfrak{P}}$ for z, namely, $z = y' + \mathfrak{m}_{\mathfrak{P}}$. Note that for $\sigma \in G \setminus D(\mathfrak{P})$, we have $\mathfrak{P} \neq \sigma^{-1}\mathfrak{P}$. By the WAT, $\exists y \in$ F s.t.

1. $\upsilon_{\mathfrak{P}}(y - y') > 0$; and
2. $\upsilon_{\sigma^{-1}\mathfrak{P}}(y) > 0 \quad \forall \sigma \in G \setminus D(\mathfrak{P})$.

As $\upsilon_{\mathfrak{P}}(y') \geq 0$ and $\upsilon_{\mathfrak{P}}(y - y') > 0$ we have that $\upsilon_{\mathfrak{P}}(y) \geq 0$, namely, $y \in \mathcal{O}_{\mathfrak{P}}$.

Item (2) above implies Item (2) of the claim since $\upsilon_{\mathfrak{P}}(\sigma y) = \upsilon_{\sigma^{-1}\mathfrak{P}}(y)$.

As for Item (1), for $\sigma \in D(\mathfrak{P})$,

$$\upsilon_{\mathfrak{P}}(\sigma y) = \upsilon_{\sigma^{-1}\mathfrak{P}}(y) = \upsilon_{\mathfrak{P}}(y) \geq 0.$$

To conclude the proof, by Item (1),

$$y + \mathfrak{m}_{\mathfrak{P}} = y' + \mathfrak{m}_{\mathfrak{P}} = z.$$

Residue fields in normal extensions

Proof. (of Theorem 16)

Going back to the proof of Theorem 16, we take $z \in \mathsf{F}_{\mathfrak{P}}$ and show that all of its $\mathsf{E}_{\mathfrak{p}}$-conjugates are in $\mathsf{F}_{\mathfrak{P}}$.

With $y = y(z)$ as in Claim 17, consider the polynomial

$$f(X) = \prod_{\sigma \in G} (X - \sigma y)^q = \prod_{\sigma \in G} (X^q - \sigma(y^q)) \in \mathsf{F}[X].$$

Since $\sigma y \in \mathcal{O}_{\mathfrak{P}}$ for all $\sigma \in G$,

$$f(X) \in \mathcal{O}_{\mathfrak{P}}[X].$$

Looking at the right expression, and since $y^q \in \mathsf{E}_s$, we have that $f(X) \in \mathsf{E}_s[X]$.

Observe that the coefficients of $f(X)$ are fixed by G and so, in fact, $f(X) \in \mathsf{E}_s^G[X] = \mathsf{E}[X]$. Thus,

$$f(X) \in (\mathsf{E} \cap \mathcal{O}_{\mathfrak{P}})[X] = \mathcal{O}_{\mathfrak{p}}[X].$$

Residue fields in normal extensions

Proof.

So far,

$$f(X) = \prod_{\sigma \in G} (X - \sigma y)^q \in \mathcal{O}_{\mathfrak{p}}[X].$$

Recall that for $\sigma \in G \setminus D(\mathfrak{P})$ we have $v_{\mathfrak{P}}(\sigma y) > 0$, namely,

$$\sigma y + \mathfrak{m}_{\mathfrak{P}} = \overline{\sigma y} = 0 \quad (\text{in } \mathsf{F}_{\mathfrak{P}} = \mathcal{O}_{\mathfrak{P}}/\mathfrak{m}_{\mathfrak{P}}).$$

Thus, the reduction of $f(X) \in \mathcal{O}_{\mathfrak{P}}[X]$ to $\bar{f}(X) \in \mathsf{F}_{\mathfrak{P}}[X]$ is

$$\begin{aligned}\bar{f}(X) &= \prod_{\sigma \in G} (X - \overline{\sigma y})^q \\ &= X^{q|G \setminus D(\mathfrak{P})|} \prod_{\sigma \in D(\mathfrak{P})} (X - \overline{\sigma y})^q \in \mathsf{E}_{\mathfrak{p}}[X].\end{aligned}$$

Residue fields in normal extensions

Proof.

We conclude that the polynomial

$$g(X) = \prod_{\sigma \in D(\mathfrak{P})} (X - \overline{\sigma y})^q \in \mathsf{E}_{\mathfrak{p}}[X]$$

has all its roots in $\mathsf{F}_{\mathfrak{P}}$ as indeed $\sigma y \in \mathcal{O}_{\mathfrak{P}}$.

Now, taking $\sigma = \mathrm{id} \in D(\mathfrak{P})$, we see that

$$g(z) = g(\bar{y}) = 0.$$

Thus, the minimal polynomial of z over $\mathsf{E}_{\mathfrak{p}}$ divides $g(X)$.

We conclude that all $\mathsf{E}_{\mathfrak{p}}$-conjugates of z are in $\mathsf{F}_{\mathfrak{P}}$, and so $\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}}$ is normal. □

Residue fields in normal extensions

Recall

$$D(\mathfrak{P}) = \{\sigma \in \mathrm{Aut}(\mathsf{F}/\mathsf{E}) \mid \sigma\mathfrak{P} = \mathfrak{P}\}.$$

Consider the map

$$\begin{aligned} \psi : D(\mathfrak{P}) &\to \mathrm{Aut}(\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}}) \\ \sigma &\mapsto \bar{\sigma} \end{aligned}$$

where $\bar{\sigma}\bar{x} = \overline{\sigma x}$ for all $x \in \mathcal{O}_{\mathfrak{P}}$ (namely, $\bar{\sigma}(x + \mathfrak{m}_{\mathfrak{P}}) = \sigma x + \mathfrak{m}_{\mathfrak{P}}$). As $\sigma \in D(\mathfrak{P})$ we have that

$$x \in \mathcal{O}_{\mathfrak{P}} \quad \Longrightarrow \quad \sigma x \in \mathcal{O}_{\sigma\mathfrak{P}} = \mathcal{O}_{\mathfrak{P}}.$$

$\bar{\sigma}$ acts as the identity on $\mathsf{E}_{\mathfrak{p}}$. Indeed, take $x \in \mathsf{E}$, then

$$\bar{\sigma}(x + \mathfrak{m}_{\mathfrak{P}}) = \sigma x + \mathfrak{m}_{\mathfrak{P}} = x + \mathfrak{m}_{\mathfrak{P}}.$$

It is easy to check that $\bar{\sigma}$ is indeed an automorphism.

Residue fields in normal extensions

Theorem 18

ψ is an epiomorphism

Proof.

We first show that ψ is a group homomorphism. Take $\sigma, \tau \in D(\mathfrak{P})$. We wish to prove that $\psi(\sigma\tau) = \psi(\sigma)\psi(\tau)$. To this end, take $x \in \mathcal{O}_{\mathfrak{P}}$.

We have that

$$\begin{aligned}\psi(\sigma)\psi(\tau)(x + \mathfrak{m}_{\mathfrak{P}}) &= \psi(\sigma)(\tau x + \mathfrak{m}_{\mathfrak{P}}) \\ &= \sigma(\tau x) + \mathfrak{m}_{\mathfrak{P}} \\ &= (\sigma\tau)x + \mathfrak{m}_{\mathfrak{P}} \\ &= \psi(\sigma\tau)(x + \mathfrak{m}_{\mathfrak{P}}).\end{aligned}$$

Residue fields in normal extensions

Proof.

We turn to show that ψ is onto.

Let N be the fixed field of $\mathrm{Aut}(F_{\mathfrak{P}}/E_{\mathfrak{p}})$. By Lemma 11, we know that $N/E_{\mathfrak{p}}$ is purely inseparable and that $F_{\mathfrak{P}}/N$ is Galois.

$$\begin{tikzcd}
F_{\mathfrak{P}} \arrow[d, dash, "\text{Galois}"'] & \\
N \arrow[d, dash, "\text{purely inseparable}"'] & \text{fixed field of } \mathrm{Aut}(F_{\mathfrak{P}}/E_{\mathfrak{p}}) \\
E_{\mathfrak{p}} &
\end{tikzcd}$$

Residue fields in normal extensions

Proof.

As $\mathrm{F}_{\mathfrak{P}}/\mathrm{N}$ is Galois and finite, by the primitive element theorem,

$$\exists z \in \mathrm{F}_{\mathfrak{P}} \quad \text{s.t.} \quad \mathrm{F}_{\mathfrak{P}} = \mathrm{N}(z).$$

As in the proof of Theorem 16, we can find $y \in \mathcal{O}_{\mathfrak{P}}$ s.t. $\bar{y} = z$ and that

$$g(X) = \prod_{\sigma \in D(\mathfrak{P})} (X - \overline{\sigma y})^q \in \mathrm{E}_{\mathfrak{p}}[X].$$

Recall that $g(z) = 0$.

Take $\tau \in \mathrm{Aut}(\mathrm{F}_{\mathfrak{P}}/\mathrm{E}_{\mathfrak{p}})$ and note that τz is also a root of g. Indeed,

$$0 = \tau(0) = \tau(g(z)) = g(\tau(z)).$$

Hence, $\exists \sigma \in D(\mathfrak{P})$ s.t.

$$\tau z = \overline{\sigma y} = \bar{\sigma}\bar{y} = \bar{\sigma} z.$$

Residue fields in normal extensions

Proof.

So far we wrote

$$\mathsf{F}_{\mathfrak{P}} = \mathsf{N}(z)$$

for some $z \in \mathsf{F}_{\mathfrak{P}}$, and proved that

$$\forall \tau \in \mathrm{Aut}(\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}}) \quad \exists \sigma \in D(\mathfrak{P}) \quad \text{s.t.} \quad \tau z = \bar{\sigma} z.$$

As $\bar{\sigma}, \tau \in \mathrm{Aut}(\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}})$,

$$\bar{\sigma}|_{\mathsf{N}} = \tau|_{\mathsf{N}} = \mathrm{id}_{\mathsf{N}}.$$

We conclude that $\bar{\sigma} = \tau$. Namely, $\tau = \psi(\sigma)$ for some $\sigma \in \mathcal{D}(\mathfrak{P})$, and so ψ is onto. $\square$

Overview

The inertia group

So far we saw the group epimorphism

$$\psi : D(\mathfrak{P}) \to \mathrm{Aut}(\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}}).$$

Definition 19

The kernel of ψ, denoted by $\mathrm{I}(\mathfrak{P}/\mathfrak{p})$ (or sometimes $\mathrm{I}(\mathfrak{P})$ for short) is called the inertia group of $\mathfrak{P}$.

Since ψ is an epimorphism, we have that

$$|D(\mathfrak{P})| = |\mathrm{I}(\mathfrak{P})| \cdot |\mathrm{Aut}(\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}})|. \tag{2}$$

Overview

Some equalities

Corollary 20

Assume F/E *is finite and normal. Denote* $e = e(\mathfrak{P}/\mathfrak{p})$, $f = f(\mathfrak{P}/\mathfrak{p})$. *Then,*

1. $\forall \sigma \in G \quad e(\sigma\mathfrak{P}) = e$ *and* $f(\sigma\mathfrak{P}) = f$.
2. $[\mathsf{F} : \mathsf{E}] = efr$ *where* r *is the number of prime divisors of* F *lying over* $\mathfrak{p}$.
3. $e = \frac{[\mathsf{F}:\mathsf{E}]_i}{[\mathsf{F}_{\mathfrak{P}}:\mathsf{E}_{\mathfrak{p}}]_i} \cdot |\mathrm{I}(\mathfrak{P}/\mathfrak{p})|$.
4. $ef = [\mathsf{F} : \mathsf{E}]_i \cdot |D(\mathfrak{P}/\mathfrak{p})|$.

Proof.

Item 1 follows immediately by the discussion so far, and since $\sigma\mathfrak{p} = \mathfrak{p}$.

Item 2 follows by Item 1 and by the fundamental equality.

Some equalities

Proof.

We turn to prove Item 3, namely,

$$e = \frac{[\mathsf{F} : \mathsf{E}]_i}{[\mathsf{F}_{\mathfrak{P}} : \mathsf{E}_{\mathfrak{p}}]_i} \cdot |\mathrm{I}(\mathfrak{P}/\mathfrak{p})|.$$

By Item 2, $e = \frac{[\mathsf{F}:\mathsf{E}]}{fr}$. Consider then

$$[\mathsf{F} : \mathsf{E}] = [\mathsf{F} : \mathsf{E}_s][\mathsf{E}_s : \mathsf{E}] = [\mathsf{F} : \mathsf{E}]_i \cdot |G|$$
$$f = [\mathsf{F}_{\mathfrak{P}} : \mathsf{E}_{\mathfrak{p}}] = [\mathsf{F}_{\mathfrak{P}} : \mathsf{E}_{\mathfrak{p}}]_i \cdot a,$$

where $a = [\mathsf{F}_{\mathfrak{P}} : \mathsf{E}_{\mathfrak{p}}]_s = |\mathrm{Aut}(\mathsf{F}_{\mathfrak{P}}/\mathsf{E}_{\mathfrak{p}})|$.

Using the orbit-stabilizer theorem we proved that $r = (G : D(\mathfrak{P}))$, and so

$$|G| = r|D(\mathfrak{P})| = r|\mathrm{I}(\mathfrak{P})| \cdot a,$$

where we used Equation 2.

Some equalities

Proof.

So far,

$$\begin{aligned} [\mathsf{F} : \mathsf{E}] &= [\mathsf{F} : \mathsf{E}]_i \cdot |G|. \\ f &= [\mathsf{F}_{\mathfrak{P}} : \mathsf{E}_{\mathfrak{p}}]_i \cdot a. \\ |G| &= r|D(\mathfrak{P})| = r|\mathrm{I}(\mathfrak{P})| \cdot a. \end{aligned}$$

Thus,

$$e = \frac{[\mathsf{F} : \mathsf{E}]}{fr} = \frac{[\mathsf{F} : \mathsf{E}]_i}{[\mathsf{F}_{\mathfrak{P}} : \mathsf{E}_{\mathfrak{p}}]_i} \cdot \frac{|G|}{ar} = \frac{[\mathsf{F} : \mathsf{E}]_i}{[\mathsf{F}_{\mathfrak{P}} : \mathsf{E}_{\mathfrak{p}}]_i} \cdot |\mathrm{I}(\mathfrak{P})|.$$

This proves Item 3.

Some equalities

Proof.

Recall that

$$[\mathsf{F} : \mathsf{E}] = [\mathsf{F} : \mathsf{E}]_i \cdot |G|.$$
$$|G| = r|D(\mathfrak{P})|.$$

We turn to prove Item 4, namely,

$$ef = [\mathsf{F} : \mathsf{E}]_i \cdot |D(\mathfrak{P}/\mathfrak{p})|.$$

We have that

$$ef = \frac{[\mathsf{F} : \mathsf{E}]}{r} = \frac{[\mathsf{F} : \mathsf{E}]_i \cdot |G|}{r} = \frac{[\mathsf{F} : \mathsf{E}]_i \cdot r|\mathcal{D}(\mathfrak{P})|}{r} = [\mathsf{F} : \mathsf{E}]_i \cdot |D(\mathfrak{P})|,$$

which completes the proof. $\square$

Some equalities

Corollary 21

Assume F/E *is a finite Galois extension and that* K *is a perfect field.*

Denote $e = e(\mathfrak{P}/\mathfrak{p})$, $f = f(\mathfrak{P}/\mathfrak{p})$ *and* $G = \mathrm{Gal}(\mathrm{F}/\mathrm{E})$*. Then,*

1. $\forall \sigma \in G \quad e(\sigma\mathfrak{P}) = e$ *and* $f(\sigma\mathfrak{P}) = f$.
2. $[\mathrm{F} : \mathrm{E}] = efr$ *where* r *is the number of prime divisors of* F *lying over* $\mathfrak{p}$.
3. $e = |\mathrm{I}(\mathfrak{P}/\mathfrak{p})|$.
4. $ef = |D(\mathfrak{P}/\mathfrak{p})|$.