# Finite FPT

[MSS]

Freeness - free

田

# Recap. In FPT

freeness is used in the definition of $\boxplus$

$$
\begin{tikzcd}
\mu_{a+b} & = & \mu_a \boxplus \mu_b \arrow[d] & \arrow[d] \\
G_{\mu_{a+b}}(z) \arrow[u, "\text{Stieltjes inversion formula}"] & & G_{\mu_a}(z) \arrow[d] & G_{\mu_b}(z) \arrow[d, "\text{inversion}"] \\
K_{\mu_{a+b}}(z) \arrow[u, "\text{inversion}"] & & K_{\mu_a}(z) \arrow[ll, "\text{The K-transform Theorem}"'] & K_{\mu_b}(z)
\end{tikzcd}
$$

**Problem.** The very definition of freeness is at the heart of the definition of $\boxplus$.

Freeness only appears in $\infty$-dim $\ddot\frown$

**Question.** Can we define $\boxplus$ in a freeness-free way? Namely, $\boxplus$ for finite dimensional operators with a similar K-transform theorem?

The key for doing so lies in our fun exercise:

Fun exercise

Let $(\mathcal{A}, \varphi)$ be a ncps. Let $a, b, u \in \mathcal{A}$ such that $u$ is Haar unitary that is free from $\{a, b\}$.

Prove that $a$ & $ubu^*$ are free.

Haar unitary finite dimensional operators exist. The K-transform also exists.

So...

Dethm. Let $A, B$ be $d \times d$ normal matrices. Then

$$\underset{\substack{Q \\ \text{Haar orthogonal}}}{\mathbb{E}} \chi_x(A + QBQ^*)$$

depends only on $\mathrm{Spec}(A)$ & $\mathrm{Spec}(B)$. We define $\boxplus_d$ as the operator on $\mathbb{C}^{\leq d}[x]$ satisfying

$$\chi_x(A) \boxplus_d \chi_x(B) = \underset{Q}{\mathbb{E}} \chi_x(A + QBQ^*)$$

**Def.** Let $O(d)$ denote the group of $d \times d$ orthogonal matrices. The **Haar distribution** is the unique distribution on $O(d)$ that is invariant under (left & right) multiplication by any $A \in O(d)$.

**Explicit sampling process.** To sample $U \sim O(d)$, generate a $d \times d$ matrix with independent Gaussians with mean 0 & variance 1, and apply Gram-Schmidt.

# Explicit Formula

$$p(x) \boxplus_d q(x) = \sum_{k=0}^{d} x^{d-k} (-1)^k c_k$$

$$c_k = \sum_{i+j=k} \frac{(d-i)!\,(d-j)!}{d!\,(d-k)!} a_i b_j$$

# Roots Formula

If $p(x) = \prod_{i=1}^{d} x - a_i$

$q(x) = \prod_{j=1}^{d} x - b_j$

Then,

$$p(x) \boxplus_d q(x) = \mathbb{E}_{\sigma \sim S_d} \prod_{k=1}^{d} x - a_k - b_{\sigma(k)}$$

# Revisiting The Cauchy- & The K-Transform

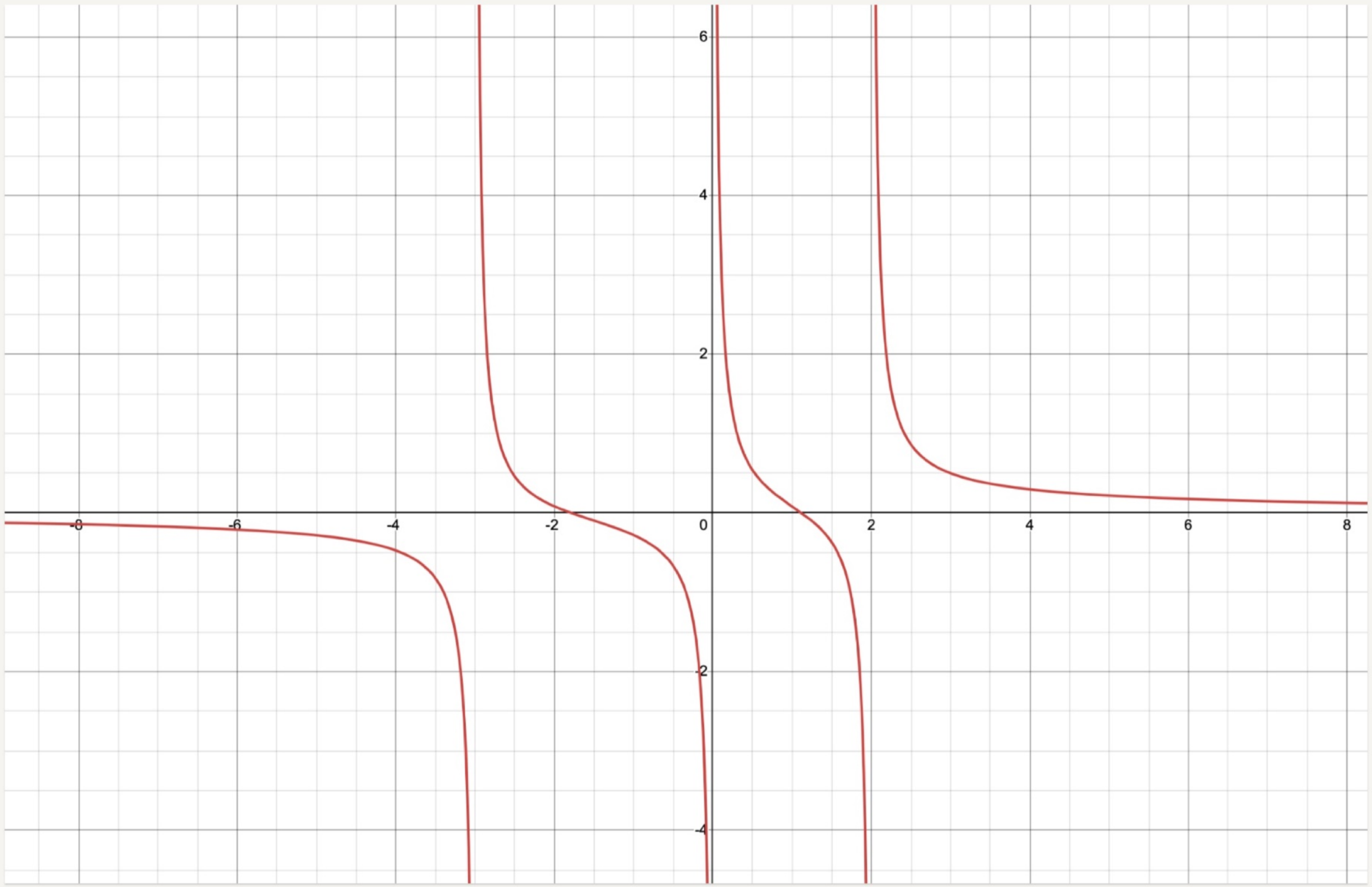
**Def.** For a polynomial $p$ with roots $\lambda_1 \geq \cdots \geq \lambda_d$, the Cauchy transform is given by

$$G_p(x) = \frac{1}{d} \sum_{i=1}^{d} \frac{1}{x - \lambda_i}$$

**Remark.**

$$G_p = G_\mu \quad \text{where} \quad \mu = \frac{1}{d} \sum_{i=1}^{d} \delta_{\lambda_i}$$

$$p(x) = (x-2)\,x\,(x+3)$$




$$G_p(x) = \frac{1}{3}\left(\frac{1}{x-2} + \frac{1}{x} + \frac{1}{x+3}\right)$$

## The K-transform.

For a real-rooted polynomial

$$p(x) = c \cdot \prod x - \lambda_i \qquad (c>0)$$

the rightmost branch $P|_{(\lambda_1, \infty)}$ has image $(0,\infty)$ & it is strictly decreasing. So we can define a max-inverse for $G_p$.

**Def.** Let $K_p : (0,\infty) \longrightarrow \mathbb{R}$ be s.t.

$$K_p(w) = \max\{x : G_p(x) = w\}$$

**Def.** Let $K_p : (0, \infty) \longrightarrow \mathbb{R}$ be s.t.

$$K_p(w) = \max \{ x : G_p(x) = w \}$$

## Remarks.

* $\forall w \in (0, \infty) \qquad K_p(w) > \text{maxroot}(p)$

* $\lim_{w \to \infty} K_p(w) = \text{maxroot}(p).$

$K_p(2) \sim 2.19$

# MSS-s K-Transform Theorem.

For $w>0$ & RRPs $p, q$ of degree $d$ having positive leading coefficients,

$$K_{p \boxplus_d q}(w) \leq K_p(w) + K_q(w) - \frac{1}{w}$$

compare to

## The K-transform Theorem.

$$K_{\mu \boxplus \nu}(z) = K_\mu(z) + K_\nu(z) - \frac{1}{z}$$

Quadrature

## Thm (MSS-S Quadrature).

Let $A, B$ be $d \times d$ symmetric matrices with $A\mathbb{1} = a\mathbb{1}$ & $B\mathbb{1} = b\mathbb{1}$. Let

$$\chi_x(A) = (x-a)\,p(x)$$

$$\chi_x(B) = (x-b)\,q(x)$$

Then

$$\mathbb{E}_{P}\, \chi_x(A + PBP^T) = (x-(a+b))\, p(x) \boxplus_{d-1} q(x)$$

$P$: uniformly random permutation matrix

$\boxplus_{d-1}$: Haar $\mathbb{O}$

Interlacing

# MSS's Interlacing Theorem.

Let $A, B$ be $d \times d$ real symmetric matrices

Assume $A\mathbb{1} = a\mathbb{1}$, $B\mathbb{1} = b\mathbb{1}$. Then

$\exists\, P_0 \in S_d$ s.t.

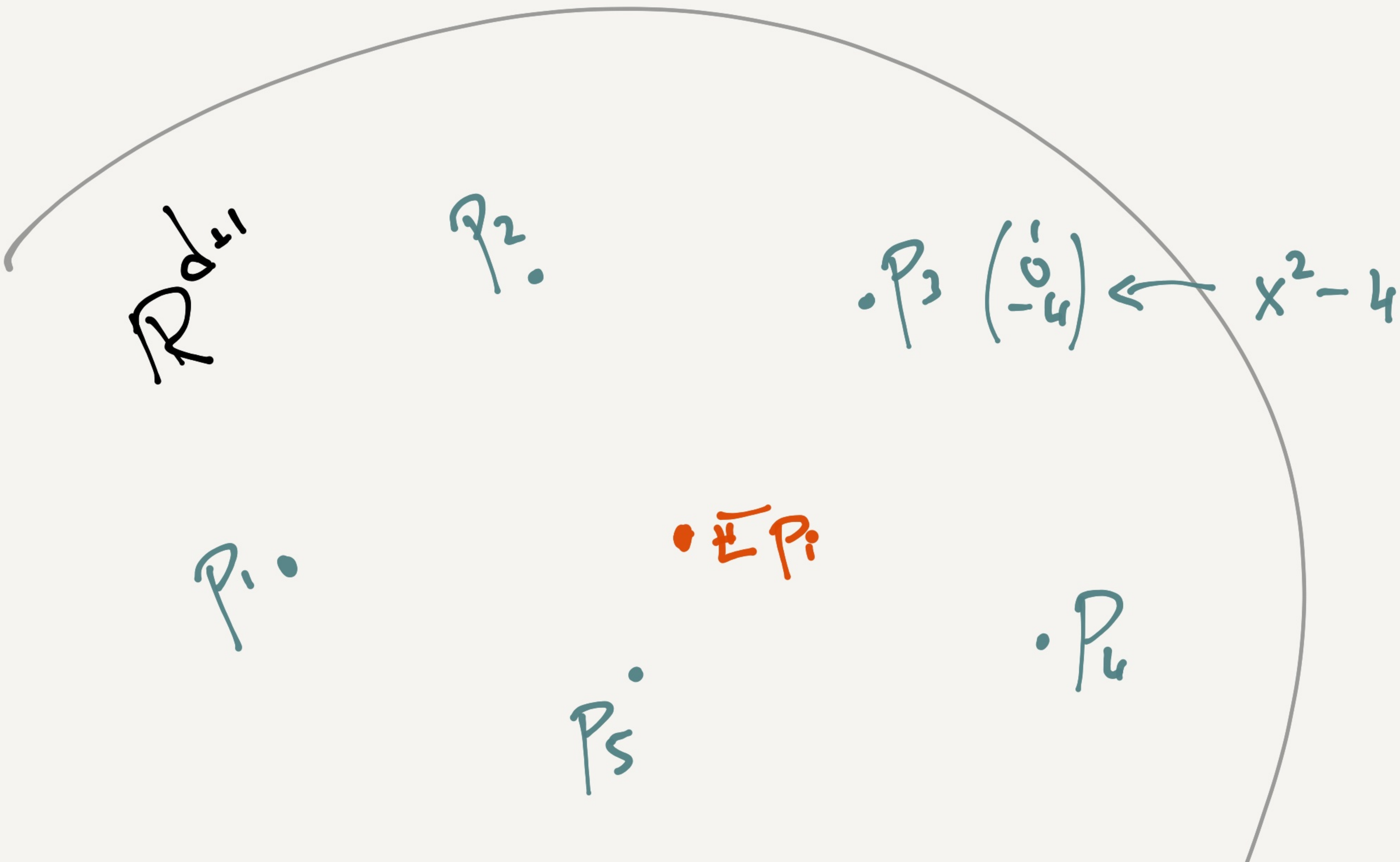
$$\text{maxroot}\ \frac{1}{x-(a+b)}\chi_x\left(A + P_0 B P_0^T\right) \leq$$

$$\text{maxroot}\ \frac{1}{x-(a+b)}\mathbb{E}_{P \sim S_d}\ \chi_x\left(A + P B P^T\right)$$

We wish to prove something like

$$\exists i \in [n] \qquad \mathrm{mr}\, p_i(x) \le \mathrm{mr} \mathop{\mathbb{E}}_{j \sim [n]} p_j(x)$$

degree d real-rooted (real) poly



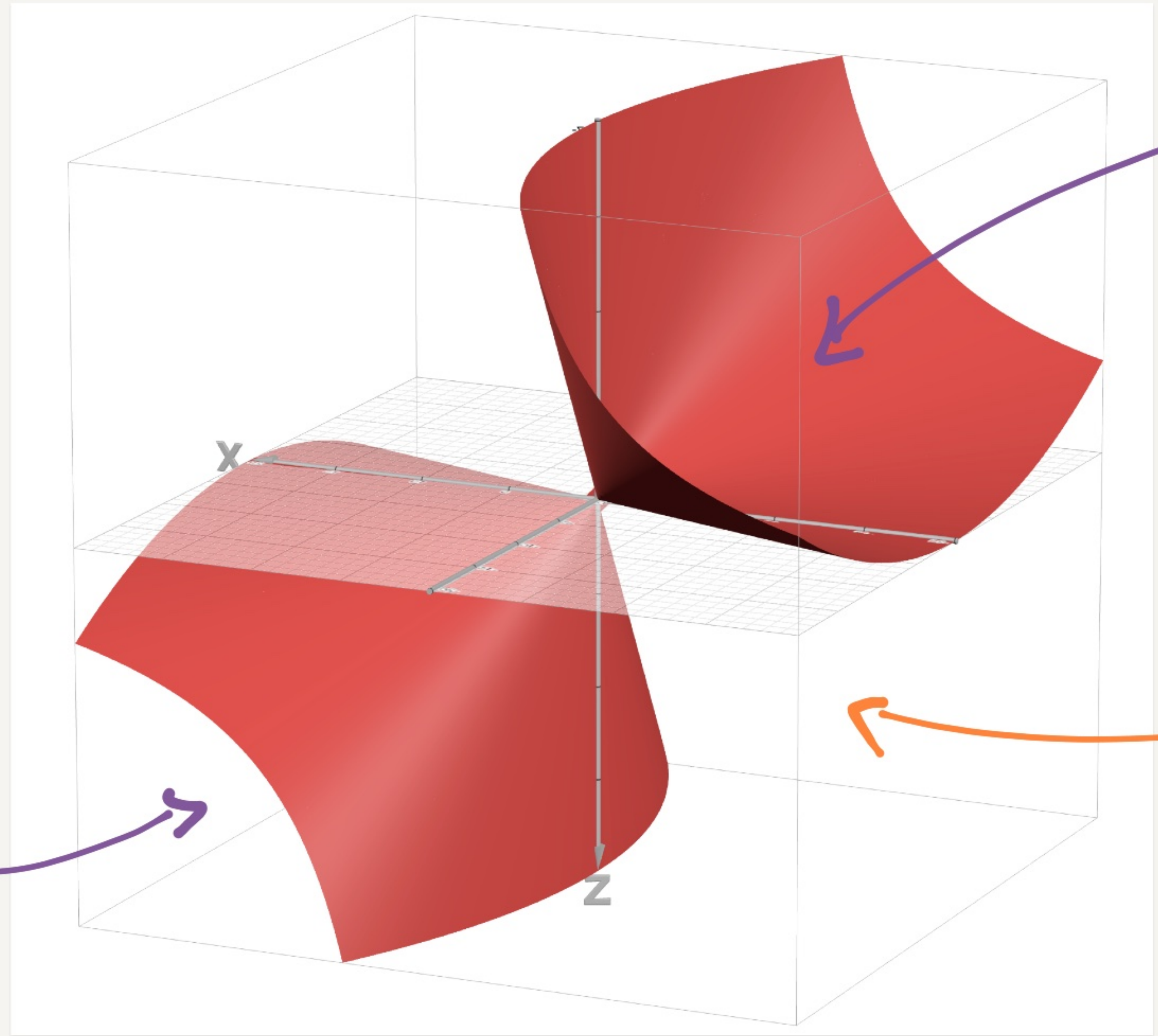
## Fact.

$$f(x) \text{ has a double-root} \iff f(x) = f'(x) = 0 \text{ has a solution}$$

$$\iff \mathrm{Res}(f, f') = 0$$

$$\mathrm{Res}(f,g) = \det \begin{pmatrix} a_n & & & -b_m & \\ \vdots & \ddots & & & \ddots \\ & & a_n & \vdots & \\ a_0 & & & & \\ & \ddots & & & \\ & & a_0 & & \end{pmatrix}$$

$\mathrm{Res}(f, f') = 0$ is the zero-set of a polynomial in coefficients space.



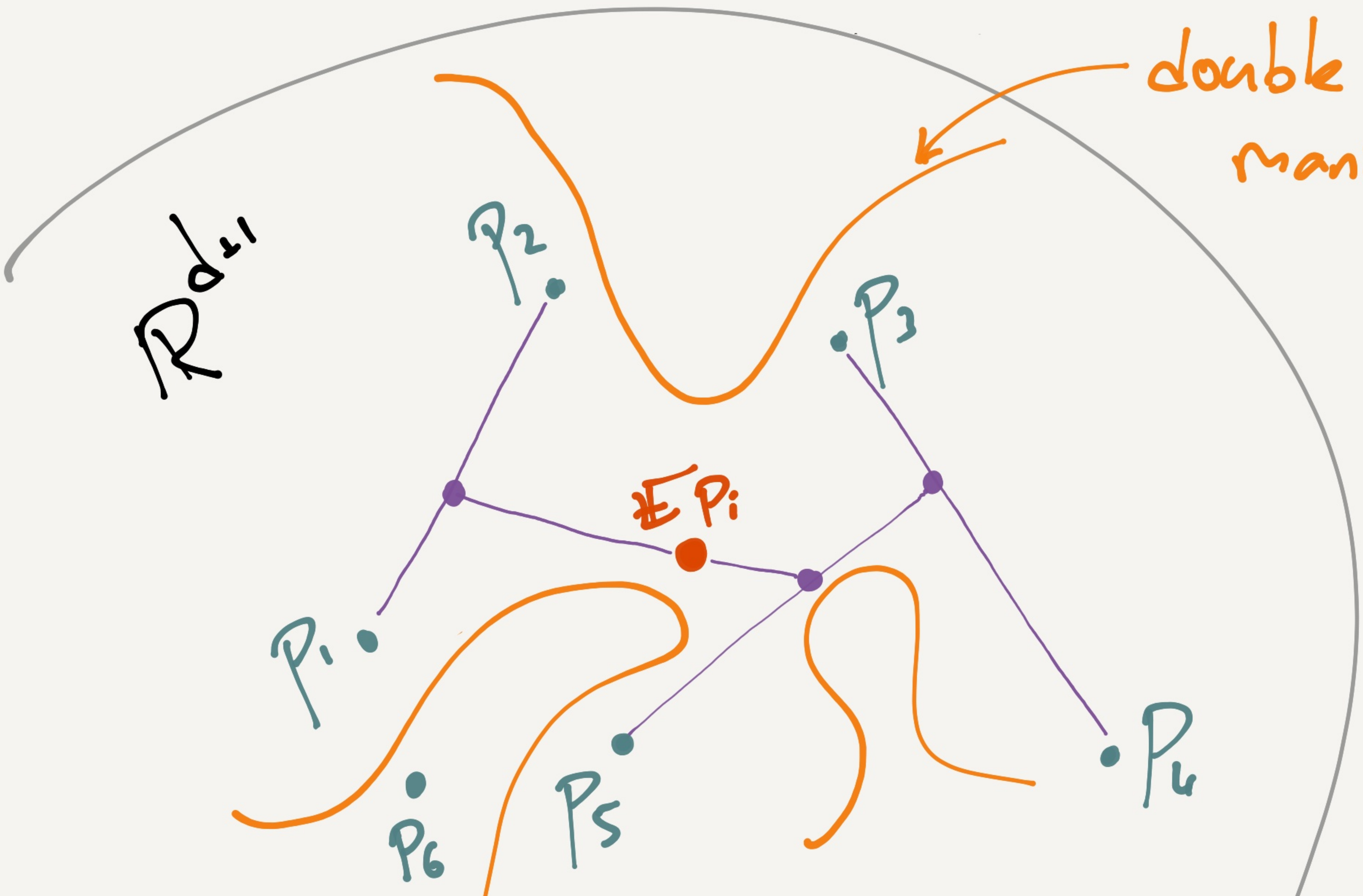

a real-rooted component

another real-rooted component

the non-real-rooted component

For $d=2$: $f(x) = ax^2+bx+c$ has a double-root $\iff b^2-4ac=0$.

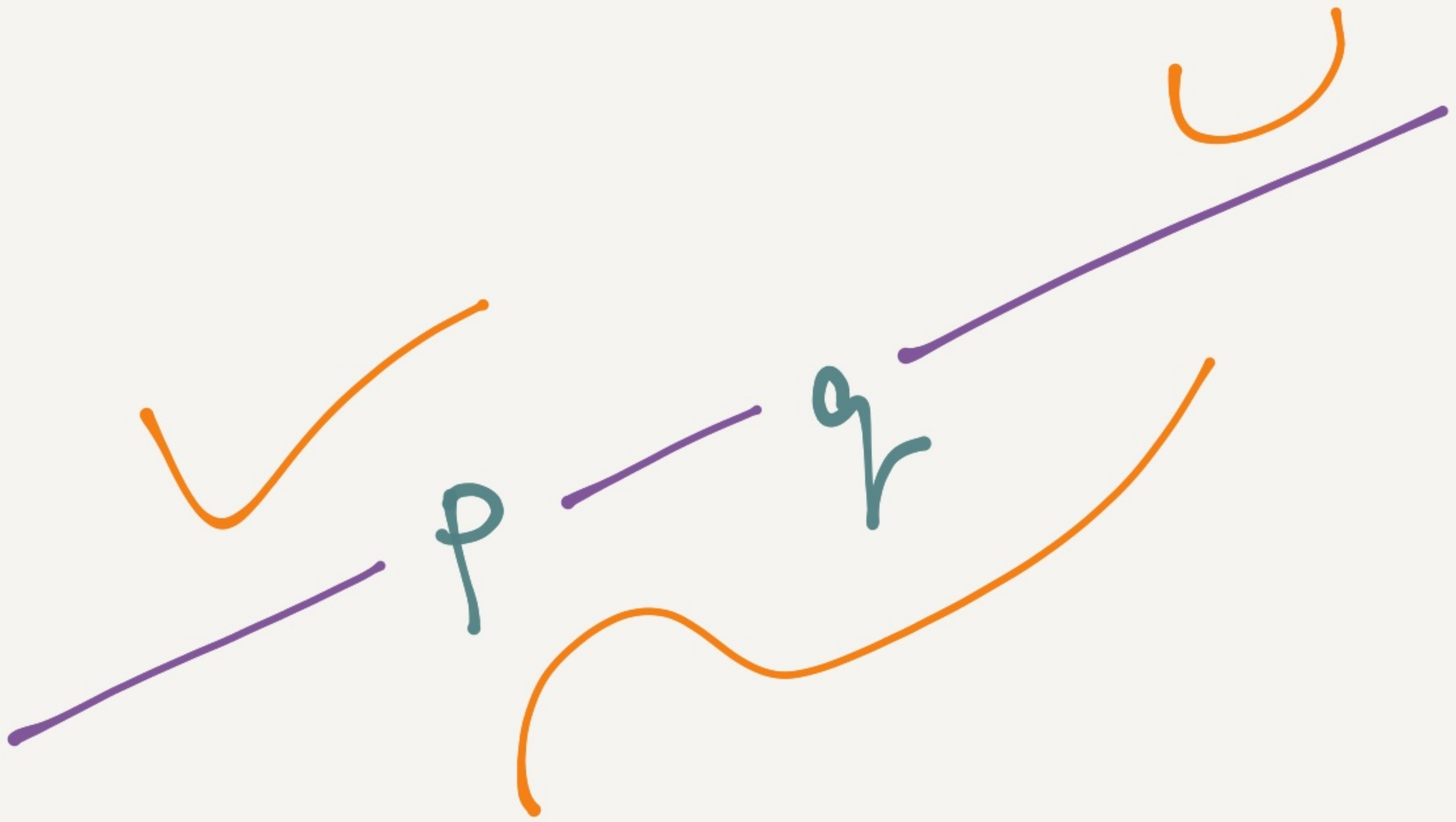
pic Def. $\{p_i\}$ is an interlacing family if:



Theorem.

The Mr function is monotone on segments contained in a RRP component

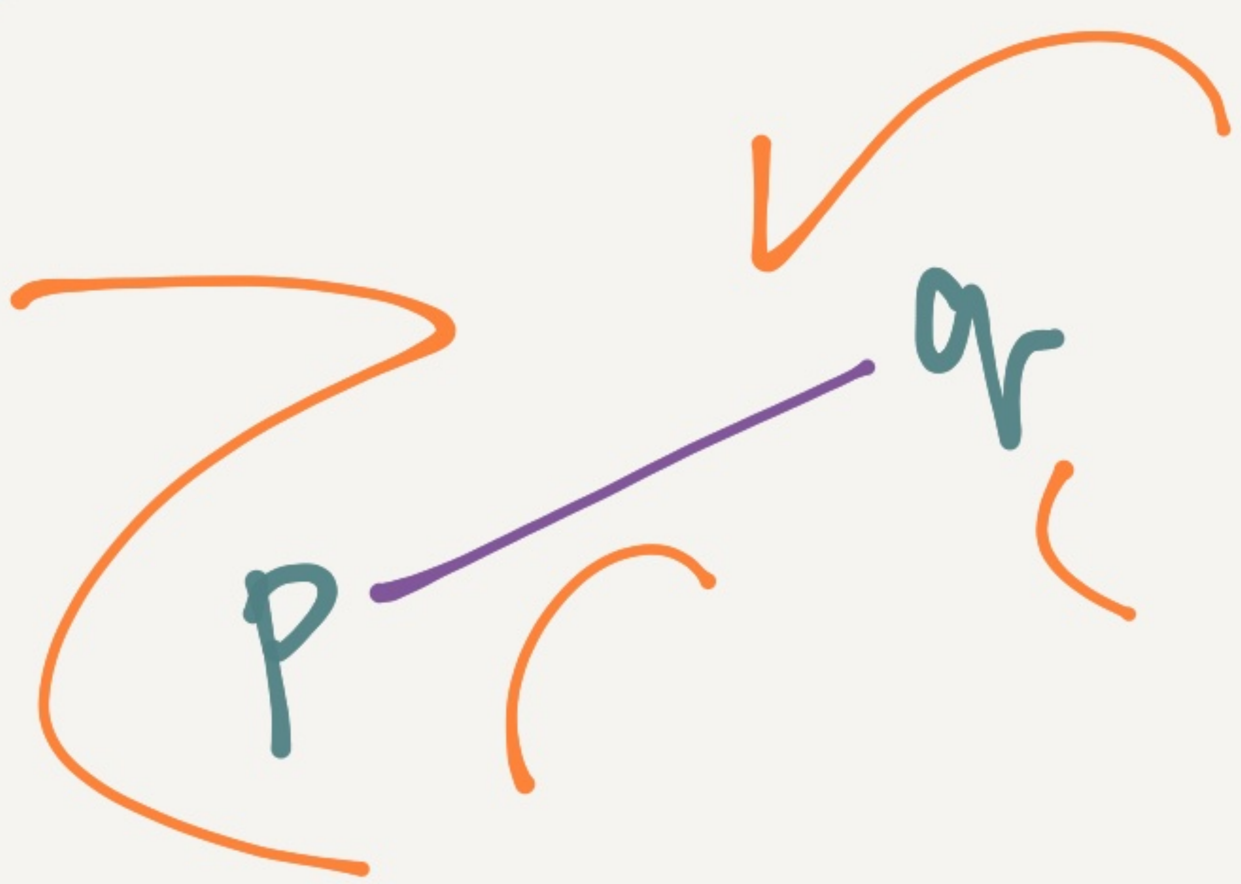
What does this has to do with interlacing?




$$S_p([p,q]) \cap \gamma = \emptyset$$

$$\Updownarrow$$

$p$ & $q$ interlace




$$[p,q] \cap \gamma = \emptyset$$

$$\Updownarrow$$

have a common interlacing

Corollary [MSS]

$\forall d \geq 3 \quad \forall n$ even

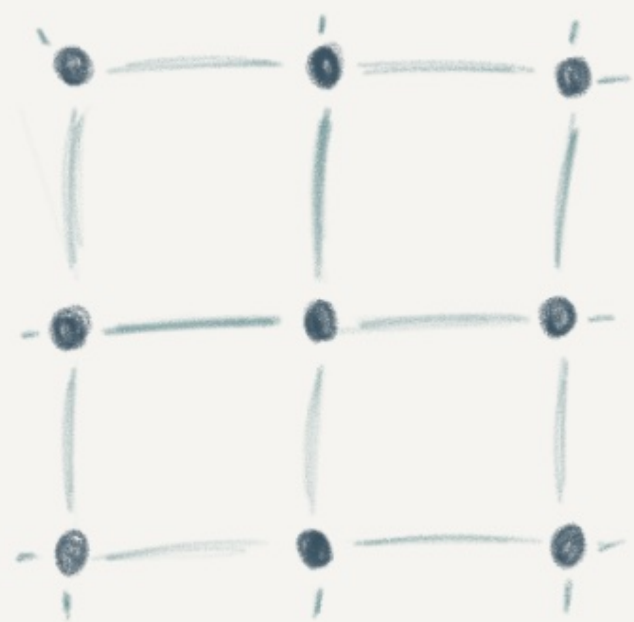
$\exists\, P_1, \ldots, P_d \in S_n$

$\lambda_2\left(P_1 M P_1^T + \cdots + P_d M P_d^T\right) \leq 2\sqrt{d-1}$

# Zig-Zag

revisited

G

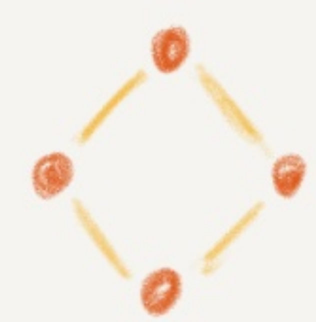


n vertices

d regular

$ⓩ$

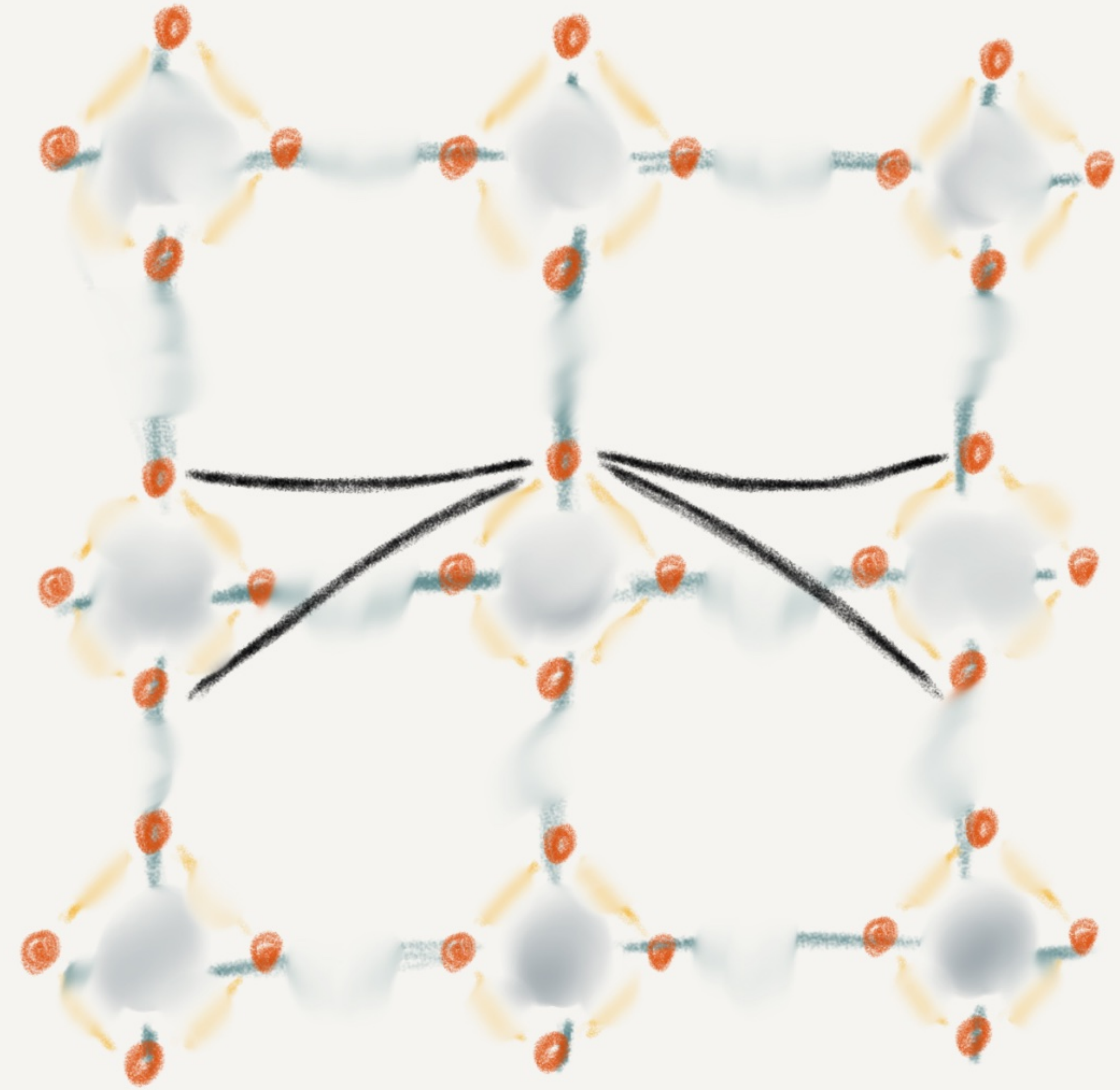
H



d vertices

c regular

$=$



$n \cdot d$ vertices

$c^2$ regular

Thm [RVW].

$$\forall G, H \qquad \omega_{G ⓩ H} \lesssim \omega_G + \omega_H$$

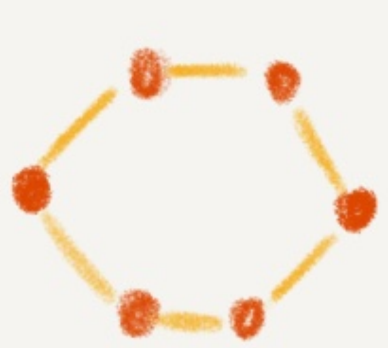
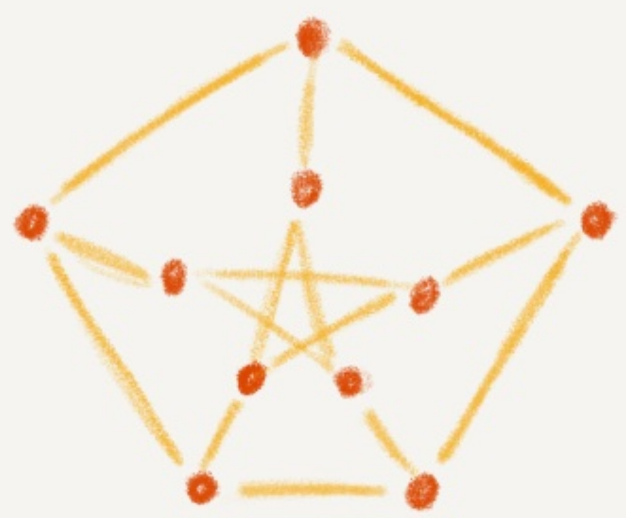
## Question.

Fix $H$. How does $\omega_{G \textcircled{z} H}$ "really" behave?

worst case

Best case

typical

| $H$ | [AB] | [RVW] | [Reality] |
| --- | --- | --- | --- |
|  | $\frac{\sqrt{3}}{2} \sim 0.86$ | $1$ | $0.96\ldots$ |
|  | $\frac{2\sqrt{8}}{9} \sim 0.63$ | $\frac{1+\sqrt{17}}{6} \sim 0.85$ | $0.79\ldots$ |

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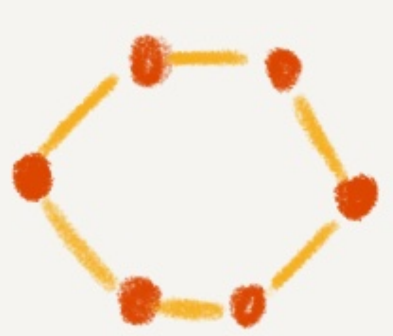
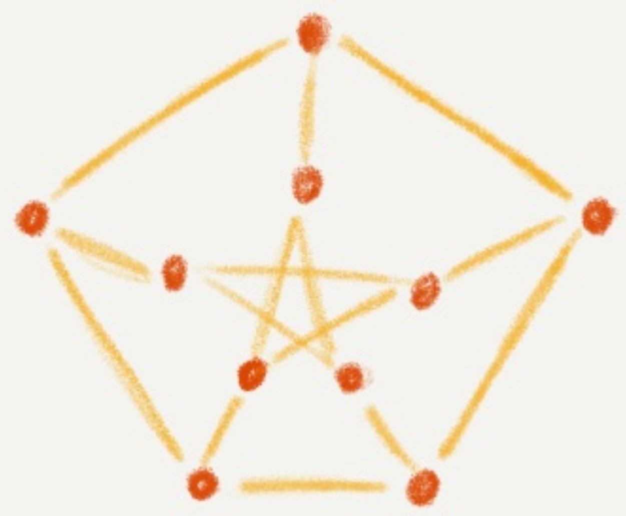
|  | $\frac{\sqrt{3}}{2} \sim 0.86$ | $1$ | $0.96\ldots$ ← | $\frac{1+\sqrt{3}}{\sqrt{8}}$ |
|  | $\frac{2\sqrt{8}}{9} \sim 0.63$ | $\frac{1+\sqrt{17}}{6} \sim 0.85$ | $0.79\ldots$ ← | not for the faint of heart |

## Question.

Fix $H$. How does $\omega_{G \circledcirc H}$ "really" behave?

worst case

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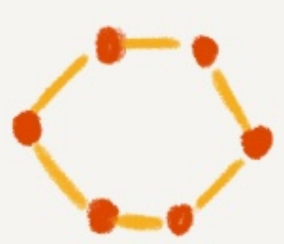
| $H$ | [AB] | [RVW] | [Reality] |
|---|---|---|---|
|  | $\frac{\sqrt{3}}{2} \sim 0.86$ | $1$ | $0.96 \leftarrow \frac{1+\sqrt{3}}{\sqrt{8}}$ |
|  | $\frac{2\sqrt{8}}{9} \sim 0.63$ | $\frac{1+\sqrt{17}}{6} \sim 0.85$ | $0.79 \leftarrow \frac{1}{3\sqrt{\frac{11}{33+\sqrt[3]{11(275-4\sqrt{11})}+\sqrt[3]{11(275+4\sqrt{11})}}}}$ |

**Question.**

Fix $H$. How does $\omega_{G \circledcirc H}$ "really" behave?

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**Main Thm.** [CCM24]

$\forall^*$ vertex-transitive $c$-regular $H$ and

$\forall$ $d$-regular $G$

$$\omega_{G \circledcirc H} \geq \min_{x > c^2} Z_H(x) - o(1)$$

where

characteristic polynomial of $H^2$

$$Z_H(x) = \frac{x}{c^2} \cdot \sqrt{1 - \frac{d \cdot h(x)}{x \cdot h'(x)}}$$

**Question.**

Fix $H$. How does $\omega_{G \circledZ H}$ "really" behave?

worst case

Best case

typical

**Main Thm.** [CCM'24]

$\forall^*$ vertex-transitive $c$-regular $H$ and $\forall G$

$$\omega_{G \circledZ H} \geq \min_{x > c^2} Z_H(x) - o(1)$$

where

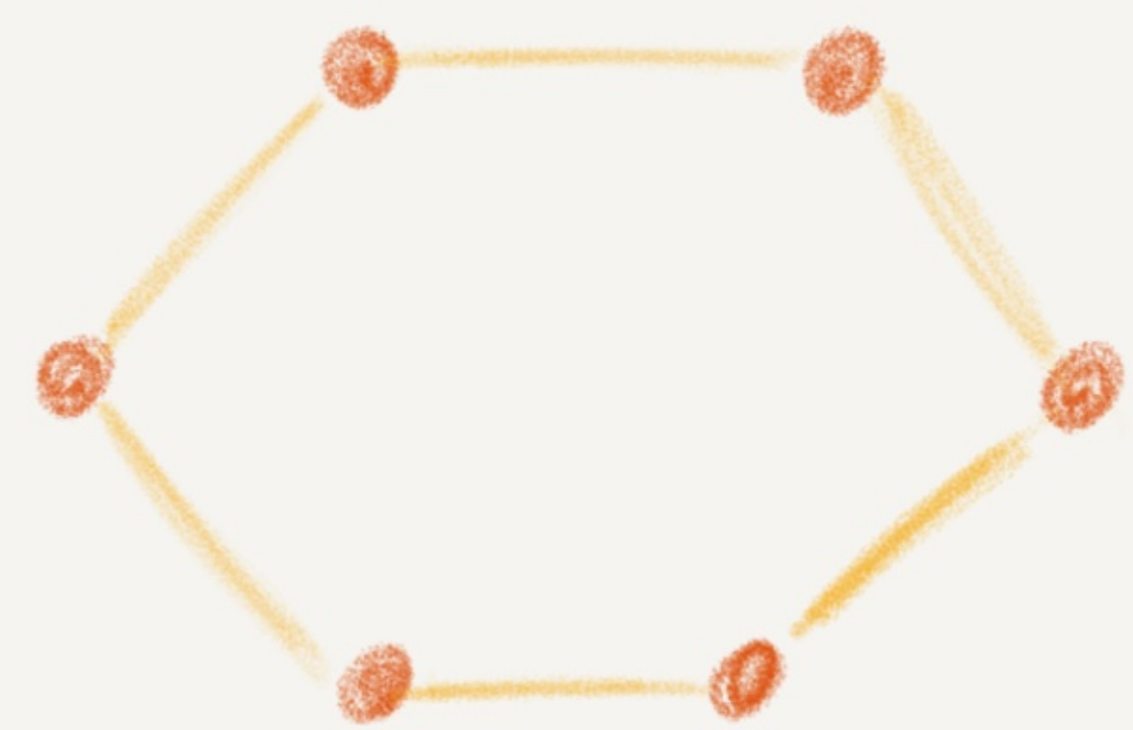
$$Z_H(x) = \frac{x}{c^2}\sqrt{1 - \frac{1}{x\, G_{H^2}(x)}}$$

The bound is "one-sided" tight:

$\forall H, n \ \exists G$ of size $n$ s.t.

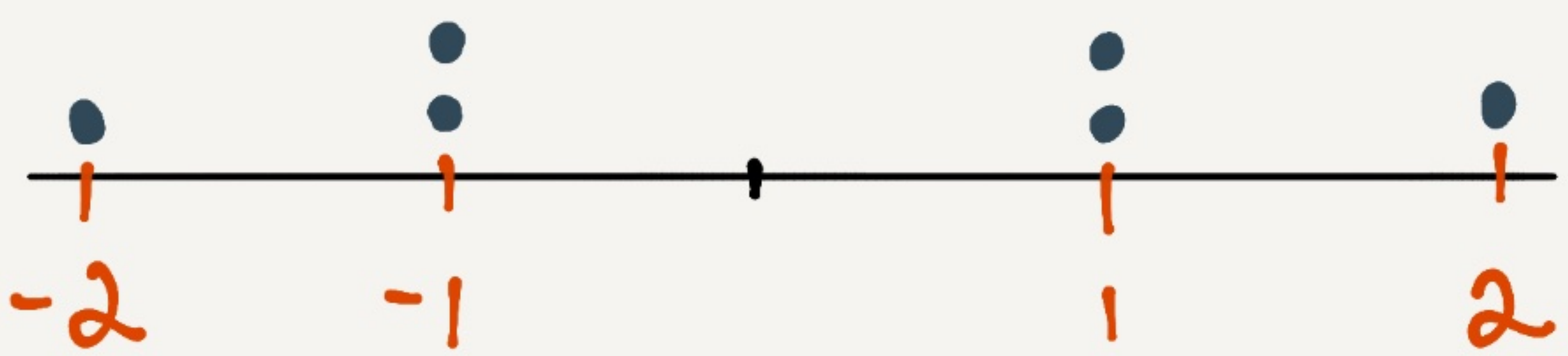
$$(\omega_2)_{G \circledZ H} \leq \min_{x > c^2} Z_H(x)$$

# Example

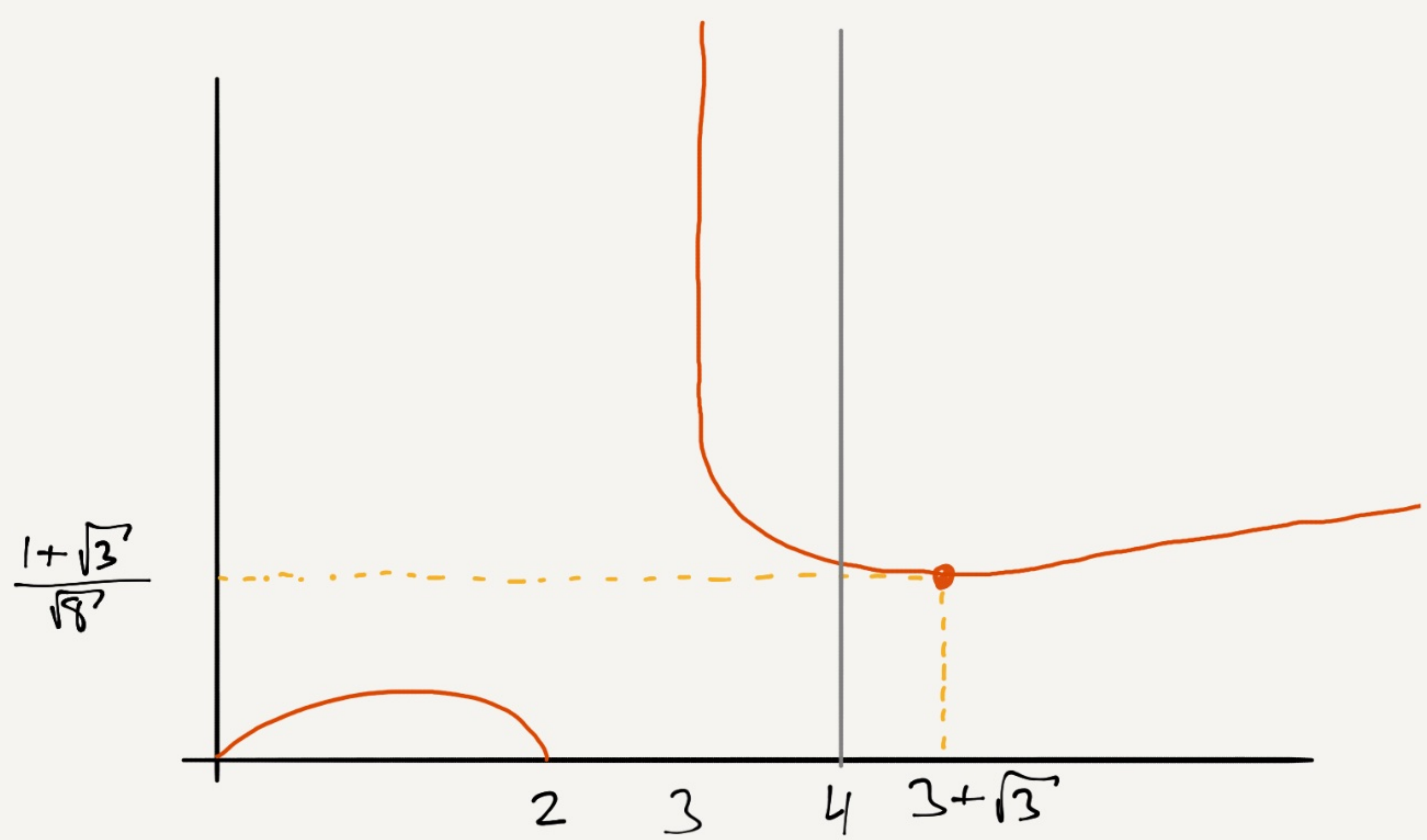


$$\left[\; Z_H(x) = \frac{x}{c^2}\cdot\sqrt{1-\frac{d\cdot h(x)}{x\cdot h'(x)}} \;\right]$$

characteristic polynomial of $H^2$




$$h(x) = (x-4)^2(x-1)^4 \quad\Longrightarrow\quad Z_{C_6}(x) = \sqrt{\frac{x(x-2)}{8(x-3)}}$$



# Example

$$\left[\; Z_H(x) = \frac{x}{c^2}\cdot\sqrt{1-\frac{d\cdot h(x)}{x\cdot h'(x)}} \;\right]$$

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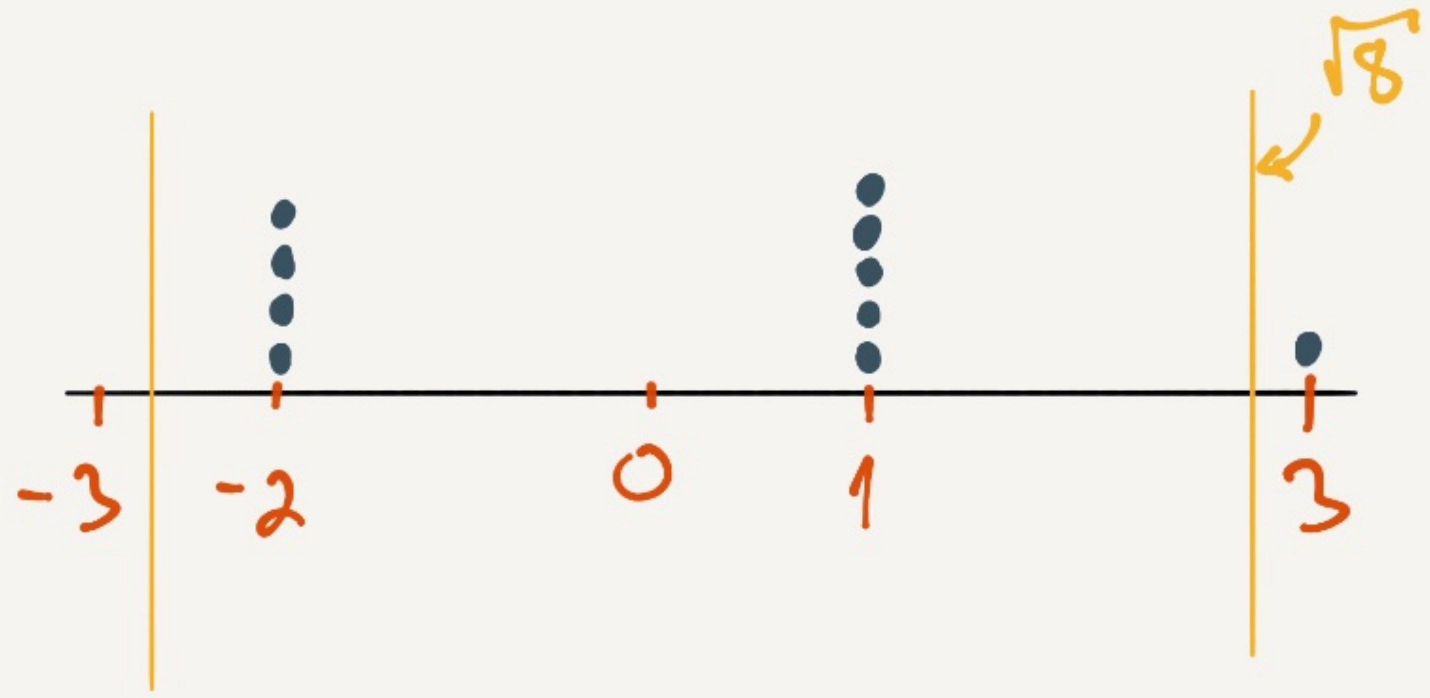
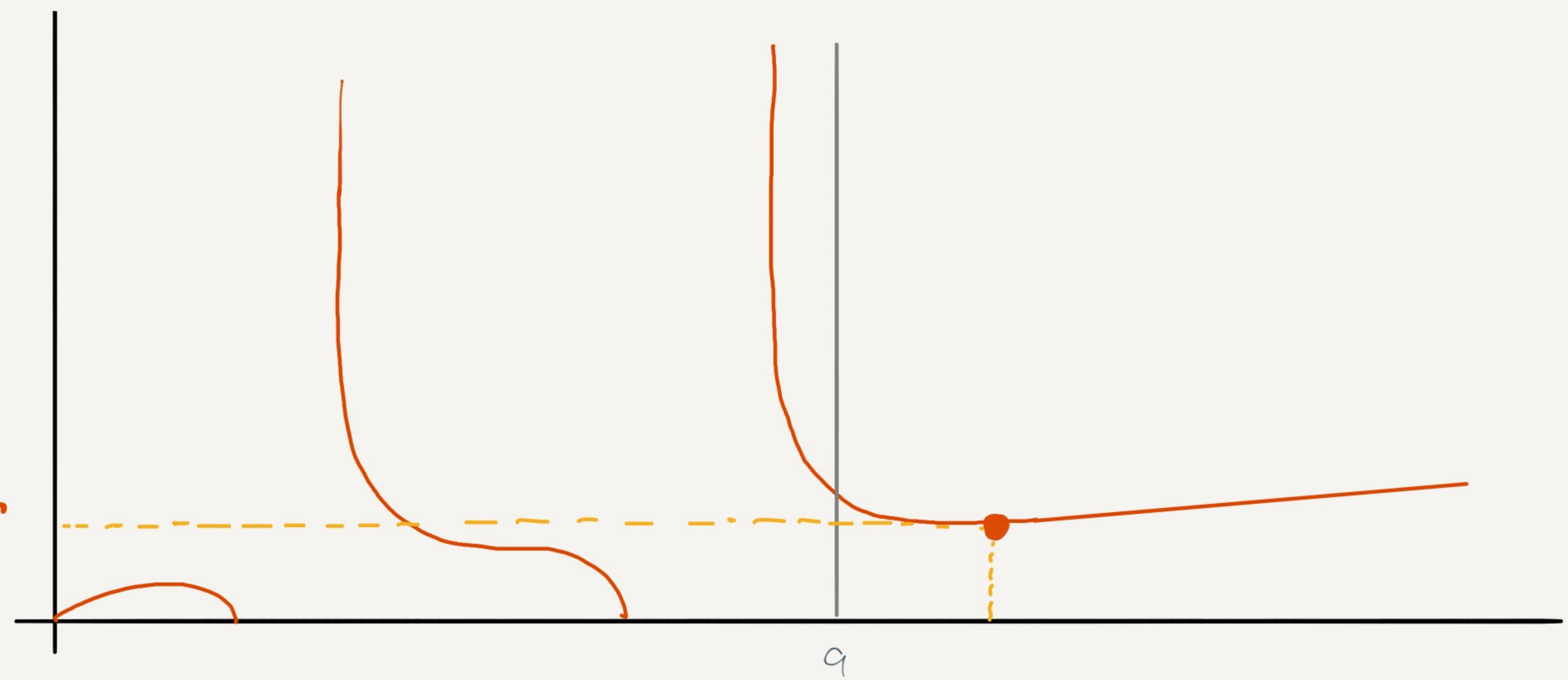



$$\Longrightarrow \quad Z_{Pet}(x) = \sqrt{\frac{x(x^2-9x+12)}{27(x^2-11x+22)}}$$

$$\frac{1}{3\sqrt{\dfrac{11}{33+\sqrt[3]{11(275-4\sqrt{11})}+\sqrt[3]{11(275+4\sqrt{11})}}}}$$



Zig Zag product $\qquad Z_H(x) = \frac{x}{c^2}\sqrt{1 - \frac{1}{x G_{H^2}(x)}}$

Replacement product $\qquad R_H(x) = \frac{1}{c+1}\left[x + \frac{\sqrt{1+4G_H(x)^2} - 1}{2G_H(x)}\right]$

Derandomized squaring $\qquad S_H(x) = \frac{1}{c}\left[x - \left(1 - \frac{1}{d}\right)\frac{1}{G_H(x)}\right]$

[RV'05] $\qquad$ [CCMP'23]

Yuval Peled

$\Longleftrightarrow$ solving $\quad (d-1)\,\chi(H)\,\ddot{\chi}(H) = (d-2)\,\dot{\chi}(H)^2$

Thanks!

Plan.

* Spectral graph theory 101
* Expander graphs → Zig Zag Product

Reingold Vadhan Wigderson 2000

"Free" union of perfect matchings

Marcus Spielman Srivastava ... 2015

Fundamental questions we can't answer.

A detour to Free Probability Theory

I ✓

∃ ONE more probability theory! Voiculescu 1985

* Freeness ≈ independence
* Central limit theorem
* Analytic machinery

II ✓

Finite FPT (interlacing, quadrature, ...)

* One-sided Ramanujan graphs
* Zig Zag revisited

III ✓