The Different

Unit 21

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Overview

1. The different
2. A technical lemma
3. Dedekind different theorem

The Different

Definition 1

Let F/L be an extension of E/K with F/E finite and separable. Let $\mathfrak{p}$ be a prime divisor of E/K with valuation ring $\mathcal{O}_{\mathfrak{p}}$ and integral closure $\mathcal{O}'_{\mathfrak{p}}$ in F. Let

$$C_{\mathfrak{p}} = t_{\mathfrak{p}} \mathcal{O}'_{\mathfrak{p}}$$

be the complementary module over $\mathcal{O}_{\mathfrak{p}}$.

We define the different exponent of $\mathfrak{P}/\mathfrak{p}$ by

$$d(\mathfrak{P}/\mathfrak{p}) = -v_{\mathfrak{P}}(t_{\mathfrak{p}}).$$

The different of F/E if defined by

$$\mathrm{Diff}(F/E) = \sum_{\mathfrak{p} \in \mathbb{P}(E)} \sum_{\mathfrak{P}/\mathfrak{p}} d(\mathfrak{P}/\mathfrak{p})\mathfrak{P}.$$

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$$\mathrm{Diff}(F/E) = \sum_{\mathfrak{p} \in \mathbb{P}(E)} \sum_{\mathfrak{P}/\mathfrak{p}} d(\mathfrak{P}/\mathfrak{p})\mathfrak{P}.$$

where $d(\mathfrak{P}/\mathfrak{p}) = -\upsilon_{\mathfrak{P}}(t_{\mathfrak{p}})$ and $C_{\mathfrak{p}} = t_{\mathfrak{p}}\mathcal{O}'_{\mathfrak{p}}$.

Some remarks are in order:

- As proved, $\upsilon_{\mathfrak{P}}(t_{\mathfrak{p}})$ does not depend on the choice of $t_{\mathfrak{p}}$ when writing $C_{\mathfrak{p}} = t_{\mathfrak{p}}\mathcal{O}'_{\mathfrak{p}}$. Thus, $d(\mathfrak{P}/\mathfrak{p})$ is well-defined.
- We further proved that $d(\mathfrak{P}/\mathfrak{p}) = 0$ for almost all $\mathfrak{P}/\mathfrak{p}$ and so Diff(F/E) is a divisor.
- $\upsilon_{\mathfrak{P}}(t_{\mathfrak{p}}) \leq 0$ and so $d(\mathfrak{P}/\mathfrak{p}) \geq 0$. Thus, $\mathrm{Diff}(F/E) \geq 0$.

The Different

Claim 2

For all $z \in F$,

$$z \in C_{\mathfrak{p}} \iff \forall \mathfrak{P}/\mathfrak{p} \quad v_{\mathfrak{P}}(z) \geq -d(\mathfrak{P}/\mathfrak{p}).$$

Proof.

$$\begin{aligned} z \in C_{\mathfrak{p}} &\iff \frac{z}{t_{\mathfrak{p}}} \in \mathcal{O}'_{\mathfrak{p}} = \bigcap_{\mathfrak{P}/\mathfrak{p}} \mathcal{O}_{\mathfrak{P}} \\ &\iff \forall \mathfrak{P}/\mathfrak{p} \quad v_{\mathfrak{P}}\left(\frac{z}{t_{\mathfrak{p}}}\right) \geq 0 \\ &\iff \forall \mathfrak{P}/\mathfrak{p} \quad v_{\mathfrak{P}}(z) \geq v_{\mathfrak{P}}(t_{\mathfrak{p}}) = -d(\mathfrak{P}/\mathfrak{p}). \end{aligned}$$

□

Overview

Technical lemma

Lemma 3

Assume F/E *is separable. Let* $\mathfrak{p} \in \mathbb{P}(\mathsf{E})$ *and* $\mathfrak{P} = \mathfrak{P}_1, \ldots, \mathfrak{P}_r \in \mathbb{P}(\mathsf{F})$ *be the prime divisors of* F *lying over* $\mathfrak{p}$*. Let*

$$\pi : \mathcal{O}_{\mathfrak{P}} \to \mathsf{F}_{\mathfrak{P}}$$

be the corresponding projective map (that can be extended to a place) which extends the projection map $\pi : \mathcal{O}_{\mathfrak{p}} \to \mathsf{E}_{\mathfrak{p}}$*.*

Let $\mathsf{F}_{\mathfrak{P},s}$ *be the separable closure of* $\mathsf{E}_{\mathfrak{p}}$ *in* $\mathsf{F}_{\mathfrak{P}}$*.*

Let $y \in \mathcal{O}'_{\mathfrak{p}}$ *be s.t.*

1. $\upsilon_{\mathfrak{P}_j}(y) > 0$ *for* $j = 2, \ldots, r$*; and*
2. $\pi(y) \in \mathsf{F}_{\mathfrak{P},s}$*.*

Then,

$$\pi\left(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)\right) = e(\mathfrak{P}/\mathfrak{p}) \cdot \mathrm{Tr}_{\mathsf{F}_{\mathfrak{P},s}/\mathsf{E}_{\mathfrak{p}}}(\pi(y)).$$

Technical lemma

Proof.

Recall that, had F/E were Galois, the ramification index would have been given by

$$e(\mathfrak{P}/\mathfrak{p}) = \frac{[\mathsf{F} : \mathsf{E}]}{[\mathsf{F}_{\mathfrak{P}} : \mathsf{E}_{\mathfrak{p}}]_i} \cdot |\mathrm{I}(\mathfrak{P}/\mathfrak{p})|.$$

However, we are not guaranteed that F/E is Galois (we only assumed F/E is separable).

To overcome this, let $\widehat{\mathsf{F}}$ be the Galois closure of F/E. Let $\widehat{\mathfrak{P}}$ be a prime divisor over $\mathfrak{P}$. We extend the projection map π to $\pi : \mathcal{O}_{\widehat{\mathfrak{P}}} \to \widehat{\mathsf{F}}_{\widehat{\mathfrak{P}}}$

We have that

$$\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y) = \sum_{i=1}^{n} \sigma_i(y)$$

where $\sigma_1, \ldots, \sigma_n : \mathsf{F} \to \widehat{\mathsf{F}}$ are the distinct E-embeddings of F into $\hat{\mathsf{F}}$.

Technical lemma

Proof.

We extend each $\sigma_i : \mathsf{F} \to \widehat{\mathsf{F}}$ to an automorphism of $\widehat{\mathsf{F}}$. Typically we will have a freedom which automorphism to pick. We will choose an automorphism arbitrarily but for the following rule: If there is an extension in $\mathcal{D}(\widehat{\mathfrak{P}}/\mathfrak{p})$ we will pick it.

We denote the extension of σ_i by $\widehat{\sigma}_i$, and assume that $\sigma_1 = \mathrm{id}_{\mathsf{F}}$ and that $\widehat{\sigma}_1 = \mathrm{id}_{\widehat{\mathsf{F}}}$ (which is consistent with the rule above).

Technical lemma

Proof.

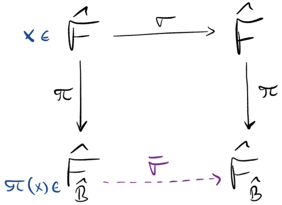
Recall that the decomposition group of $\widehat{\mathfrak{P}}$ is given by

$$\mathcal{D}(\widehat{\mathfrak{P}}/\mathfrak{p}) = \{\sigma \in \mathrm{Gal}(\widehat{\mathsf{F}}/\mathsf{E}) \mid \sigma\widehat{\mathfrak{P}} = \widehat{\mathfrak{P}}\},$$

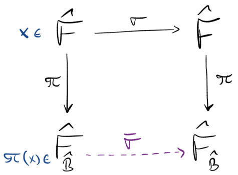
and the epimorphism

$$\psi : D(\widehat{\mathfrak{P}}/\mathfrak{p}) \to \mathrm{Aut}(\widehat{\mathsf{F}}_{\widehat{\mathfrak{P}}}/\mathsf{E}_{\mathfrak{p}})$$
$$\sigma \mapsto \bar{\sigma}$$

where $\forall x \in \widehat{\mathsf{F}} \qquad \bar{\sigma}(\pi x) = \pi(\sigma x).$



Technical lemma



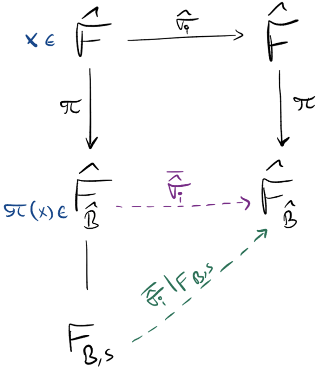
Proof.

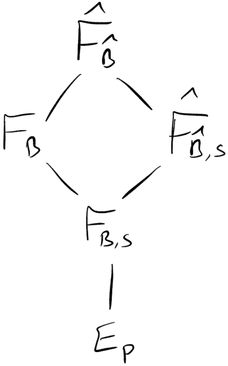
Note further that $\mathrm{Aut}(\widehat{\mathsf{F}}_{\hat{\mathfrak{P}}}/\mathsf{E}_{\mathfrak{p}}) = \mathrm{Gal}(\widehat{\mathsf{F}}_{\hat{\mathfrak{P}},s}/\mathsf{E}_{\mathfrak{p}})$ as the extension $\widehat{\mathsf{F}}_{\hat{\mathfrak{P}}}/\widehat{\mathsf{F}}_{\hat{\mathfrak{P}},s}$ is purely inseparable and so every automorphism of $\widehat{\mathsf{F}}_{\hat{\mathfrak{P}},s}/\mathsf{E}_{\mathfrak{p}}$ can be uniquely extended to an automorphism of $\widehat{\mathsf{F}}_{\hat{\mathfrak{P}}}/\mathsf{E}_{\mathfrak{p}}$.

Of course, in our case, the restriction of $\bar{\hat{\sigma}}_i$ to $\hat{\mathsf{F}}_{\hat{\mathfrak{P}},s}$ is an $\mathsf{E}_{\mathfrak{p}}$-embedding. Thus, for every $\hat{\sigma}_i \in \mathcal{D}(\hat{\mathfrak{P}}/\mathfrak{p})$ we have that $\bar{\hat{\sigma}}_i$ is an embedding over $\mathsf{E}_{\mathfrak{p}}$ of $\mathsf{F}_{\mathfrak{P},s}$ into $\hat{\mathsf{F}}_{\hat{\mathfrak{P}}}$. In particular,

$$\{\bar{\hat{\sigma}}_i(\pi(y)) \mid \hat{\sigma}_i \in \mathcal{D}(\hat{\mathfrak{P}}/\mathfrak{p})\} \subseteq \{\alpha(\pi(y)) \mid \mathsf{E}_{\mathfrak{p}} \text{ embedding } \alpha : \mathsf{F}_{\mathfrak{P},s} \to \hat{F}_{\hat{\mathfrak{P}}}\}.$$

Technical lemma





Technical lemma

Proof.

We turn to prove the other direction, namely,

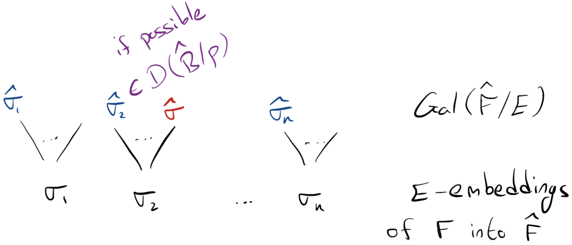
$$\{\bar{\hat{\sigma}}_i(\pi(y)) \mid \hat{\sigma}_i \in \mathcal{D}(\hat{\mathfrak{P}}/\mathfrak{p})\} \supseteq \{\alpha(\pi(y)) \mid \mathsf{E}_\mathfrak{p} \text{ embedding } \alpha : \mathsf{F}_{\mathfrak{P},s} \to \hat{\mathsf{F}}_{\hat{\mathfrak{P}}}\}.$$

Indeed, take $\alpha : \mathsf{F}_{\mathfrak{P},s} \to \hat{\mathsf{F}}_{\hat{\mathfrak{P}}}$ an embedding over $\mathsf{E}_\mathfrak{p}$, and extend it to an automorphism $\hat{\alpha}$ of $\hat{\mathsf{F}}_{\hat{\mathfrak{P}}}$ over $\mathsf{E}_\mathfrak{p}$ (this is an automorphism as $\hat{\mathsf{F}}_{\hat{\mathfrak{P}}}$ is normal).

Recall that the map $\psi : D(\hat{\mathfrak{P}}/\mathfrak{p}) \to \mathrm{Aut}(\hat{\mathsf{F}}_{\hat{\mathfrak{P}}}/\mathsf{E}_\mathfrak{p})$ is onto. Thus, there exists $\hat{\sigma} \in D(\hat{\mathfrak{P}}/\mathfrak{p}) \leq \mathrm{Aut}(\hat{\mathsf{F}}/\mathsf{E})$ such that $\bar{\hat{\sigma}} = \psi(\hat{\sigma}) = \hat{\alpha}$.

Had $\hat{\sigma} = \hat{\sigma}_i$ for some i we would have been done. However, it is not necessarily the case that $\hat{\sigma}$ is one of the extensions of σ_i as there is freedom in how to choose an extension $\hat{\sigma}_i$ of σ_i.

Technical lemma



Proof.

Nonetheless, we will show how to tweak $\hat{\sigma}$ so to get the result. Indeed,

$$\pi(\hat{\sigma} y) = \bar{\hat{\sigma}}(\pi(y)) = \hat{\alpha}(\pi(y)).$$

Now, per our assumption, $\pi(y) \in \mathrm{F}_{\mathfrak{P},s}$. As $\hat{\alpha}$ extends α from $\mathrm{F}_{\mathfrak{P},s}$ to $\hat{\mathrm{F}}_{\hat{\mathfrak{P}}}$, we have that

$$\pi(\hat{\sigma} y) = \alpha(\pi(y)).$$

Technical lemma

Proof.

Now, $\hat{\sigma}$ restricted to F is some σ_i, namely,

$$\hat{\sigma}|_{\mathsf{F}} = \sigma_i|_{\mathsf{F}} \quad \text{and so} \quad (\hat{\sigma}^{-1}\sigma_i)|_{\mathsf{F}} = \mathrm{id}_{\mathsf{F}}.$$

Thus, $\hat{\sigma}^{-1}\sigma_i = \tau \in \mathrm{Aut}(\hat{\mathsf{F}}/\mathsf{F})$. Namely, $\hat{\sigma} = \sigma_i\tau$.

Since $\hat{\sigma} \in \mathcal{D}(\hat{\mathfrak{P}}/\mathfrak{p})$ extends σ_i it is also the case that $\hat{\sigma}_i \in \mathcal{D}(\hat{\mathfrak{P}}/\mathfrak{p})$ as by our rule, if there is an extension of σ_i in $\mathcal{D}(\hat{\mathfrak{P}}/\mathfrak{p})$ then we pick such extension.

As $\tau|_{\mathsf{F}} = \mathrm{id}_{\mathsf{F}}$ we have that

$$\pi(\sigma_i(y)) = \pi(\sigma_i(\tau y)) = \pi((\sigma_i\tau)y) = \pi(\hat{\sigma}(y)) = \bar{\sigma}(\pi(y)) = \alpha(\pi(y)).$$

Technical lemma

Proof.

Consider now the case that $\hat{\sigma}_i \hat{\mathfrak{P}} \neq \hat{\mathfrak{P}}$ or, equivalently, $\hat{\sigma}_i{}^{-1}\hat{\mathfrak{P}} \neq \hat{\mathfrak{P}}$. Let $\mathfrak{P}'$ be the prime divisor of F that is under $\hat{\sigma}_i{}^{-1}\mathfrak{P}$. We claim that $\mathfrak{P}' \neq \mathfrak{P}$.

Otherwise, there exists $\tau \in \mathrm{Aut}(\hat{\mathsf{F}}/\mathsf{F})$ such that

$$\tau\hat{\mathfrak{P}} = \hat{\mathfrak{P}}' \quad \text{and so} \quad \hat{\sigma}_i \tau \in \mathcal{D}(\hat{\mathfrak{P}}/\mathfrak{P}) \subseteq \mathcal{D}(\hat{\mathfrak{P}}/\mathfrak{p}).$$

But $\tau|_{\mathsf{F}} = \mathrm{id}_{\mathsf{F}}$ and so $\hat{\sigma}_i \tau|_{\mathsf{F}} = \hat{\sigma}_i|_{\mathsf{F}}$.

This stands in contradiction to our assumption $\hat{\sigma}_i \hat{\mathfrak{P}} \neq \hat{\mathfrak{P}}$. Indeed, by our rule and under this assumption, when choosing $\hat{\sigma}_i$ as the extension of σ_i there was no choice of an extension in $\mathcal{D}(\hat{\mathfrak{P}}/\mathfrak{p})$.

Since $\mathfrak{P}' \neq \mathfrak{P}$ we have, per our assumption, that $\upsilon_{\mathfrak{P}'}(y) > 0$ and so $\upsilon_{\hat{\sigma}_i{}^{-1}\hat{\mathfrak{P}}}(y) > 0$, and so

$$\upsilon_{\hat{\mathfrak{P}}}(\sigma_i y) = \upsilon_{\hat{\sigma}_i{}^{-1}\hat{\mathfrak{P}}}(y) > 0 \quad \Longrightarrow \quad \hat{\sigma}_i y \in \mathcal{O}_{\hat{\mathfrak{P}}} \text{ and } \pi(\hat{\sigma}_i y) = 0.$$

Technical lemma

Proof.

Denote $\widehat{\mathcal{D}} = \mathcal{D}(\widehat{\mathfrak{P}}/\mathfrak{p})$ and

$$A = \{\alpha : \mathsf{F}_{\mathfrak{P},s} \to \widehat{\mathsf{F}}_{\hat{\mathfrak{P}}} \mid \alpha \text{ that is an } \mathsf{E}_{\mathfrak{p}}\text{-embedding}\}.$$

Recall that for i such that $\hat{\sigma}_i \notin \hat{\mathcal{D}}$ we have that $\pi(\hat{\sigma}_i(y)) = 0$, and so

$$\begin{aligned} \pi(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)) &= \sum_{i=1}^{n} \pi(\sigma_i(y)) = \sum_{i=1}^{n} \pi(\widehat{\sigma}_i(y)) \\ &= \sum_{\sigma_i \in \hat{\mathcal{D}}} \pi(\hat{\sigma}_i(y)) = \sum_{\sigma_i \in \hat{\mathcal{D}}} (\bar{\hat{\sigma}}_i(\pi(y))) \\ &= \sum_{\alpha \in A} |\{i \mid \hat{\sigma}_i \in \widehat{\mathcal{D}}, \bar{\hat{\sigma}}_i = \alpha\}| \cdot \alpha(\pi(y)). \end{aligned}$$

Technical lemma

Proof.

To summarize,

$$\pi(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)) = \sum_{\alpha \in A} |\{i \mid \hat{\sigma}_i \in \hat{\mathcal{D}}, \bar{\hat{\sigma}}_i = \alpha\}| \cdot \alpha(\pi(y)).$$

but

$$|\{i \mid \hat{\sigma}_i \in \hat{\mathcal{D}}, \bar{\hat{\sigma}}_i = \alpha\}| = e(\mathfrak{P}/\mathfrak{p}),$$

and so

$$\pi(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)) = e(\mathfrak{P}/\mathfrak{p}) \cdot \sum_{\alpha \in A} \alpha(\pi(y)) = e(\mathfrak{P}/\mathfrak{p}) \cdot \mathrm{Tr}_{\mathsf{F}_{\mathfrak{P},s}/\mathsf{E}_{\mathfrak{p}}}(\pi(y)).$$

□

Overview

Dedekind Different Theorem

Theorem 4 (Dedekind Different Theorem)

Let F/L *be a finite separable extension of* E/K. *Let* $\mathfrak{p} \in \mathbb{P}(E)$ *and* $\mathfrak{P} \in \mathbb{P}(F)$ *lying over* $\mathfrak{p}$. *Then,*

1. $d(\mathfrak{P}/\mathfrak{p}) \geq e(\mathfrak{P}/\mathfrak{p}) - 1$; *and*
2. $d(\mathfrak{P}/\mathfrak{p}) = e(\mathfrak{P}/\mathfrak{p}) - 1 \iff \operatorname{char} K \nmid e(\mathfrak{P}/\mathfrak{p})$.

Corollary 5

With the above notations,

$$d(\mathfrak{P}/\mathfrak{p}) = 0 \iff e(\mathfrak{P}/\mathfrak{p}) = 1$$

In particular, for almost all $\mathfrak{p}, \mathfrak{P}/\mathfrak{p}$ *we have that* $e(\mathfrak{P}/\mathfrak{p}) = 1$.

The proof of the corollary is straightforward.

Dedekind Different Theorem

Proof.

Let $\mathfrak{P} = \mathfrak{P}_1, \ldots, \mathfrak{P}_n$ be all prime divisors lying over $\mathfrak{p}$. By the WAT $\exists z \in \mathsf{F}$ s.t.

$$\forall i \in [n] \quad \upsilon_{\mathfrak{P}_i}(z) = 1 - e(\mathfrak{P}_i/\mathfrak{p}).$$

To prove the first item, it suffices to show that

$$z \in \mathsf{C}_{\mathfrak{p}} = t_{\mathfrak{p}} \mathcal{O}'_{\mathfrak{p}} = t_{\mathfrak{p}} \bigcap_{i=1}^{n} \mathcal{O}_{\mathfrak{P}_i}.$$

Indeed, if this is the case then

$$1 - e(\mathfrak{P}_i/\mathfrak{p}) = \upsilon_{\mathfrak{P}_i}(z) \geq \upsilon_{\mathfrak{P}_i}(t_{\mathfrak{p}}) = -d(\mathfrak{P}_i/\mathfrak{p}).$$

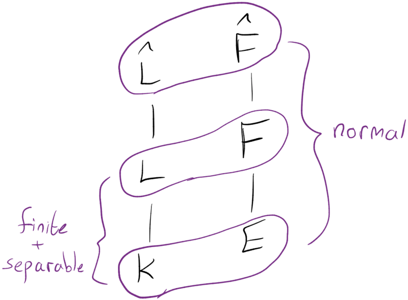
In particular, $d(\mathfrak{P}/\mathfrak{p}) \geq e(\mathfrak{P}/\mathfrak{p}) - 1$.

Dedekind Different Theorem

Proof.

Let $\widehat{F}$ be the Galois closure of F/E and $\widehat{L}$ be the algebraic closure of K in $\widehat{F}$.

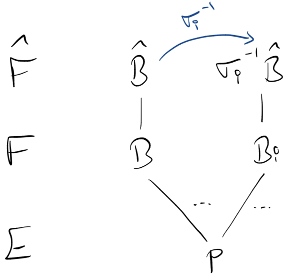
Denote $n = [F : E]$ and let $\sigma_1, \ldots, \sigma_n$ be the E-embeddings of F in $\widehat{F}$. n such embeddings exist since F/E is separable and since $\widehat{F}$ is the Galois closure of F/E. We extend each σ_i to an automorphism of $\widehat{F}$ over E.



Dedekind Different Theorem

Proof.

Choose $\widehat{\mathfrak{P}}$ over $\mathfrak{P}$ and consider the prime divisor $\sigma_i^{-1}\widehat{\mathfrak{P}}$. As $\sigma_i^{-1} \in \mathrm{Gal}(\widehat{F}/E)$, $\sigma_i^{-1}\widehat{\mathfrak{P}}$ lies over $\mathfrak{p}$. Let $\mathfrak{P}_i \in \mathbb{P}(F)$ be the prime divisor of F between $\mathfrak{p}$ and $\sigma_i^{-1}\widehat{\mathfrak{P}}$.



Dedekind Different Theorem

Proof.

Take $x \in \mathcal{O}'_{\mathfrak{p}}$ and note that $\sigma_i(x) \in \mathcal{O}'_{\mathfrak{p}}$. Thus, $\upsilon_{\widehat{\mathfrak{P}}}(\sigma_i(x)) \geq 0$. Now,

$$\begin{aligned} \upsilon_{\widehat{\mathfrak{P}}}(\sigma_i(zx)) &= \upsilon_{\widehat{\mathfrak{P}}}(\sigma_i(z)\sigma_i(x)) \\ &= \upsilon_{\widehat{\mathfrak{P}}}(\sigma_i(z)) + \upsilon_{\widehat{\mathfrak{P}}}(\sigma_i(x)) \\ &\geq \upsilon_{\widehat{\mathfrak{P}}}(\sigma_i(z)) \\ &= \upsilon_{\sigma_i^{-1}\widehat{\mathfrak{P}}}(z). \end{aligned}$$

As $z \in$ F we have that

$$\begin{aligned} \upsilon_{\sigma_i^{-1}\widehat{\mathfrak{P}}}(z) &= e(\sigma_i^{-1}\widehat{\mathfrak{P}}/\mathfrak{P}_i)\upsilon_{\mathfrak{P}_i}(z) \\ &= e(\sigma_i^{-1}\widehat{\mathfrak{P}}/\mathfrak{P}_i)(1 - e(\mathfrak{P}_i/\mathfrak{p})) \\ &> -e(\sigma_i^{-1}\widehat{\mathfrak{P}}/\mathfrak{P}_i) \cdot e(\mathfrak{P}_i/\mathfrak{p}) \\ &= -e(\sigma_i^{-1}\widehat{\mathfrak{P}}/\mathfrak{p}) = -e(\widehat{\mathfrak{P}}/\mathfrak{p}). \end{aligned}$$

Dedekind Different Theorem

Proof.

So far we have that

$$\forall i \quad \upsilon_{\widehat{\mathfrak{P}}}(\sigma_i(zx)) > -e(\widehat{\mathfrak{P}}/\mathfrak{p}).$$

So,

$$\begin{aligned} e(\widehat{\mathfrak{P}}/\mathfrak{p})\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(zx)) &= \upsilon_{\widehat{\mathfrak{P}}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(zx)) \\ &= \upsilon_{\widehat{\mathfrak{P}}}\left(\sum_{i=1}^{n}\sigma_i(zx)\right) \\ &\geq \min_i \upsilon_{\widehat{\mathfrak{P}}}(\sigma_i(zx)) \\ &> -e(\widehat{\mathfrak{P}}/\mathfrak{p}). \end{aligned}$$

Therefore, $\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(zx)) > -1$ or, equivalently, $\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(zx)) \geq 0$ and so $\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(zx) \in \mathcal{O}_{\mathfrak{p}}$. Thus, by the definition of the complementary module, $z \in \mathsf{C}_{\mathfrak{p}}$.

Dedekind Different Theorem

Proof.

We turn to prove the second item. Assume first that

$$\mathrm{charK} \nmid e(\mathfrak{P}/\mathfrak{p}) \triangleq e.$$

Since the first item implies that $d(\mathfrak{P}/\mathfrak{p}) \geq e - 1$ and we wish to show equality, it suffices to prove that $d(\mathfrak{P}/\mathfrak{p}) < e$.

As before let $\mathfrak{P} = \mathfrak{P}_1, \ldots, \mathfrak{P}_r$ be all the distinct prime divisors of F lying over $\mathfrak{p}$. Denote $e_i = e(\mathfrak{P}_i/\mathfrak{p})$.

Take $t_\mathfrak{p} \in \mathrm{F}$ s.t. $\mathrm{C}_\mathfrak{p} = t_\mathfrak{p}\mathcal{O}'_\mathfrak{p}$. Then,

$$\upsilon_{\mathfrak{P}_i}(t_\mathfrak{p}) = -d(\mathfrak{P}_i/\mathfrak{p}).$$

Dedekind Different Theorem

Claim 6

$\exists y \in \mathcal{O}'_{\mathfrak{p}}$ s.t.

1. $\upsilon_{\mathfrak{P}}(y) = 0$;
2. $\upsilon_{\mathfrak{P}_i}(y) \geq \max(1, e_i + \upsilon_{\mathfrak{P}_i}(t_{\mathfrak{p}}))$ for $i = 2, \ldots, r$; and
3. $\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)) = 0$.

Proof.

Let $\mathsf{F}_{\mathfrak{P},s}$ be the separable closure of $\mathsf{E}_{\mathfrak{p}}$ in $\mathsf{F}_{\mathfrak{P}}$. As $\mathsf{F}_{\mathfrak{P},s}/\mathsf{E}_{\mathfrak{p}}$ is separable,

$$\exists \bar{y}_0 \in \mathsf{F}_{\mathfrak{P},s} \quad \text{s.t.} \quad \mathrm{Tr}_{\mathsf{F}_{\mathfrak{P},s}/\mathsf{E}_{\mathfrak{p}}}(\bar{y}_0) \neq 0.$$

As $\bar{y}_0 \in \mathsf{F}_{\mathfrak{P}}$, $\exists y_0 \in \mathcal{O}_{\mathfrak{P}}$ s.t. $\pi(y_0) = \bar{y}_0$.

By WAT, $\exists y \in \mathsf{F}$ s.t. $\upsilon_{\mathfrak{P}}(y - y_0) > 0$ and for which Item 2 holds, namely,

$$\upsilon_{\mathfrak{P}_i}(y) \geq \max(1, e_i + \upsilon_{\mathfrak{P}_i}(t_{\mathfrak{p}})) \quad i = 2, \ldots, r.$$

Dedekind Different Theorem

Proof.

We turn to show that Item 1 holds. Indeed,

$$v_{\mathfrak{P}}(y) = v_{\mathfrak{P}}(y - y_0 + y_0).$$

Since $v_{\mathfrak{P}}(y - y_0) > 0$, showing that $v_{\mathfrak{P}}(y) = 0$ would follow if $v_{\mathfrak{P}}(y_0) = 0$.

To see that this is the case, if $v_{\mathfrak{P}}(y_0) > 0$ then

$$\bar{y}_0 = \pi(y_0) = 0,$$

and so

$$\mathrm{Tr}_{\mathsf{F}_{\mathfrak{P},s}/\mathsf{E}_{\mathfrak{p}}}(\bar{y}_0) = 0,$$

in contradiction to our choice of $\bar{y}_0$.

Dedekind Different Theorem

Proof.

To conclude the proof, we prove Item 3, namely that

$$\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(y)) = 0.$$

To this end we wish to apply Lemma 3 and so we first make sure the hypothesis of the lemma are satisfied.

Hypothesis 1 of Lemma 3 follows by Item 2.

Hypothesis 2 ($\pi(y) \in F_{\mathfrak{P},s}$) follows since $\upsilon_{\mathfrak{P}}(y - y_0) > 0$ and so

$$\pi(y) = \pi(y_0) = \bar{y}_0 \in F_{\mathfrak{P},s}.$$

The only thing left to show is that $y \in \mathcal{O}'_{\mathfrak{p}}$ which follows since $y \in \mathcal{O}_{\mathfrak{P}_i}$ for all i.

Dedekind Different Theorem

Proof.

Applying Lemma 3 we conclude that

$$\pi\left(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)\right) = e \cdot \mathrm{Tr}_{\mathsf{F}_{\mathfrak{P},s}/\mathsf{E}_{\mathfrak{p}}}(\pi(y)),$$

where note that the equation is over K.

Now, $\pi(y) = \pi(y_0) = \bar{y}_0$ and so

$$\mathrm{Tr}_{\mathsf{F}_{\mathfrak{P},s}/\mathsf{E}_{\mathfrak{p}}}(\pi(y)) = \mathrm{Tr}_{\mathsf{F}_{\mathfrak{P},s}/\mathsf{E}_{\mathfrak{p}}}(\bar{y}_0) \neq 0.$$

As we assume $\mathrm{char}\mathsf{K} \nmid e$, we have that $e \neq 0$ in K and so, overall,

$$\pi\left(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)\right) \neq 0.$$

Hence,

$$v_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)) = 0,$$

proving Claim 6. □

Dedekind Different Theorem

Proof.

Going back to the proof of Theorem 4, take $x \in \mathsf{E}$ s.t. $\upsilon_{\mathfrak{p}}(x) = -1$ (and so $\upsilon_{\mathfrak{P}_i}(x) = -e_i$).

Since we found in Claim 6 $y \in \mathsf{F}$ s.t.

1. $\upsilon_{\mathfrak{P}}(y) = 0$;
2. $\upsilon_{\mathfrak{P}_i}(y) \geq \max(1, e_i + \upsilon_{\mathfrak{P}_i}(t_{\mathfrak{p}}))$ for $i = 2, \ldots, r$; and
3. $\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)) = 0$,

we get that

1. $\upsilon_{\mathfrak{P}}(xy) = \upsilon_{\mathfrak{P}}(x) + \upsilon_{\mathfrak{P}}(y) = -e$;
2. For $i = 2, \ldots, r$,
$$\upsilon_{\mathfrak{P}_i}(xy) \geq \max(1, e_i + \upsilon_{\mathfrak{P}_i}(t_{\mathfrak{p}})) - e_i \geq \upsilon_{\mathfrak{P}_i}(t_{\mathfrak{p}}) = -d(\mathfrak{P}_i/\mathfrak{p});$$
3. $\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(xy)) = \upsilon_{\mathfrak{p}}(x\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)) = \upsilon_{\mathfrak{p}}(x) + \upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y)) = -1.$

Dedekind Different Theorem

Proof.

Denote $y' = xy$. Then,

1. $v_{\mathfrak{P}}(y') = -e$;
2. $v_{\mathfrak{P}_i}(y') \geq v_{\mathfrak{P}_i}(t_{\mathfrak{p}}) = -d(\mathfrak{P}_i/\mathfrak{p})$ for $i = 2, \ldots, r$; and
3. $v_{\mathfrak{p}}(\mathrm{Tr}_{\mathrm{F/E}}(y')) = -1$.

By (3), $y' \notin \mathrm{C}_{\mathfrak{p}}$ (as $1 \in \mathcal{O}'_{\mathfrak{p}}$ and for y' to be in $\mathrm{C}_{\mathfrak{p}}$ we must have $\mathrm{Tr}_{\mathrm{F/E}}(zy') \in \mathcal{O}_{\mathfrak{p}}$ for all $z \in \mathcal{O}'_{\mathfrak{p}}$.) Recall that

$$y' \in \mathrm{C}_{\mathfrak{p}} = t_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}} \iff v_{\mathfrak{P}_i}(y') \geq -d(\mathfrak{P}_i/\mathfrak{p}) \quad \text{for } i = 1, 2, \ldots, r.$$

By (2) we therefore must have

$$v_{\mathfrak{P}}(y') < -d(\mathfrak{P}/\mathfrak{p}).$$

(1) then implies that $e > d(\mathfrak{P}/\mathfrak{p})$ which concludes the proof for

$$\mathrm{char}K \nmid e \implies d(\mathfrak{P}/\mathfrak{p}) = e(\mathfrak{P}/\mathfrak{p}) - 1.$$

Dedekind Different Theorem

Proof.

To complete the proof we need to show that

$$\operatorname{char} \mathsf{K} \mid e \quad \Longrightarrow \quad d(\mathfrak{P}/\mathfrak{p}) \geq e(\mathfrak{P}/\mathfrak{p}).$$

To prove this we prove the following claim.

Claim 7

$\exists y \in \mathcal{O}'_{\mathfrak{p}}$ s.t. $\forall z \in \mathcal{O}'_{\mathfrak{p}}$ the following holds:

1. $\upsilon_{\mathfrak{P}}(y) = 0$;
2. $\upsilon_{\mathfrak{P}}(yz) \geq 0$;
3. $\upsilon_{\mathfrak{P}_i}(yz) > 0$ for $i = 2, \ldots, r$; and
4. $\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(yz)) > 0$.

Dedekind Different Theorem

$\exists y \in \mathcal{O}'_{\mathfrak{p}} \quad \forall z \in \mathcal{O}'_{\mathfrak{p}}$

1. $\upsilon_{\mathfrak{P}}(y) = 0$;
2. $\upsilon_{\mathfrak{P}}(yz) \geq 0$;
3. $\upsilon_{\mathfrak{P}_i}(yz) > 0$ for $i = 2, \ldots, r$; and
4. $\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathrm{F}/\mathrm{E}}(yz)) > 0$.

Proof.

By WAT, $\exists y \in \mathrm{F}$ s.t.

$$\upsilon_{\mathfrak{P}}(y) = 0 \quad \text{and} \quad \upsilon_{\mathfrak{P}_i}(y) > 0 \ \text{for} \ i > 1.$$

In particular, $y \in \mathcal{O}'_{\mathfrak{p}}$ and items 1,2, and 3 hold.

As for Item 4, denote

$$q = [\mathrm{F}_{\mathfrak{P}} : \mathrm{F}_{\mathfrak{P},s}].$$

Dedekind Different Theorem

Proof.

By Lemma 3 applied to $y' = (yz)^q$ whose hypothesis holds, in particular,

$$\pi(y') = \pi((yz)^q) = (\pi(yz))^q \in \mathsf{F}_{\mathfrak{P},s},$$

we have that

$$\pi(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}((yz)^q)) = e \cdot \mathrm{Tr}_{\mathsf{F}_{\mathfrak{P},s}/\mathsf{E}_{\mathfrak{p}}}(\pi((yz)^q)) = 0,$$

where the last equality holds since $e = 0$ in $\mathsf{F}_{\mathfrak{P}}$.

Thus,

$$v_{\mathfrak{p}}((\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(yz))^q) = v_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}((yz)^q)) > 0,$$

and so

$$v_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(yz)) > 0.$$

Item 4 then follows, proving Claim 7.

Dedekind Different Theorem

$\exists y \in \mathcal{O}'_{\mathfrak{p}} \quad \forall z \in \mathcal{O}'_{\mathfrak{p}}$

1. $v_{\mathfrak{P}}(y) = 0$;
2. $v_{\mathfrak{P}}(yz) \geq 0$;
3. $v_{\mathfrak{P}_i}(yz) > 0$ for $i = 2, \ldots, r$; and
4. $v_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(yz)) > 0$.

Proof.

Going back to the proof of Theorem 4, by multiplying y by $x \in E$ with $v_{\mathfrak{p}}(x) = -1$ we get that for $y' = xy$ and $\forall z \in \mathcal{O}'_{\mathfrak{p}}$ it holds that

1. $v_{\mathfrak{P}}(y') = -e$;
2. $v_{\mathfrak{P}}(y'z) \geq -e$;
3. $v_{\mathfrak{P}_i}(y'z) > -e_i$ for $i = 2, \ldots, r$; and
4. $v_{\mathfrak{p}}(\mathrm{Tr}_{F/E}(y'z)) \geq 0$.

Dedekind Different Theorem

Proof.

For $y' = xy$ and $\forall z \in \mathcal{O}'_{\mathfrak{p}}$ it holds that

1. $\upsilon_{\mathfrak{P}}(y') = -e$;
2. $\upsilon_{\mathfrak{P}}(y'z) \geq -e$;
3. $\upsilon_{\mathfrak{P}_i}(y'z) > -e_i$ for $i = 2, \ldots, r$; and
4. $\upsilon_{\mathfrak{p}}(\mathrm{Tr}_{\mathsf{F}/\mathsf{E}}(y'z)) \geq 0$.

By Item 4, $y' \in \mathsf{C}_{\mathfrak{p}}$ and so

$$\upsilon_{\mathfrak{P}}(y') \geq -d(\mathfrak{P}/\mathfrak{p}).$$

The proof then follows by Item 1. □