Localization

Rings of Fractions

Localization is a commutative-algebraic tool that allows one the invert certain elements in a ring. A familiar example is the construction of $\mathbb{Q}$ from $\mathbb{Z}$ in which every nonzero element becomes invertible. One may choose to invert fewer elements. E.g., the subring $\mathbb{Z}[\frac{1}{3}]$ of $\mathbb{Q}$ that consisting of fractions $\frac{a}{3^s}$ $s \geq 0$ (which can be constructed without "going all the way" to $\mathbb{Q}$).

A second example is the ring of all fraction $\frac{a}{b}$ with $3 \nmid b$. More generally $\mathbb{Z}_p = \{ \frac{a}{b} \in \mathbb{Q} \mid p \nmid b \}$ for p prime. This is a subring of $\mathbb{Q}$ that is extremely simple in that all of the prime numbers other than p are invertible. It has only one nonzero prime ideal $-$ $\langle p \rangle$.

Definition

A domain. A set $S \subseteq A$ is multiplicative if

1) $0 \notin S$ & $1 \in S$

2) $\forall a, b \in S \quad ab \in S$

Examples

The examples from before are related to the following multiplicative sets:

* Take $0 \neq \alpha \in A$ and define $S = \{1, \alpha, \alpha^2, \ldots\}$. S is multiplicative.

* Take P prime ideal. Then, $S = A \setminus P$ is multiplicative.

Let A domain and $S \subseteq A$ multiplicative. We define the ring $S^{-1}A$ which, informally, will adjoin to A the inverses of S. First we'll define $S^{-1}A$ as a set and then endow it with a ring structure:

thought of as $\frac{a}{s}$

Consider the set $A \times S$ and the following relation: $(a,s) \sim (a',s')$ if $as' - a's = 0$.

This is indeed an equivalence relation.

Let $\frac{a}{s}$ denote the equivalence class of (a,s) in $A \times S/\sim$. Let $S^{-1}A$ denote the set of e. classes. We endow $S^{-1}A$ with a ring structure by

$$\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$$

well defined as S is multiplicative

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

$$\left(\frac{1}{1} = \text{identity} \right)$$

$S^{-1}A$ is called <u>the ring of fractions of A w.r.t S.</u>

$A \hookrightarrow S^{-1}A$

There is a natural embedding $j_S : A \hookrightarrow S^{-1}A$ $a \mapsto \frac{a}{1}$. This is indeed a ring homomorphism & mono as $\frac{a}{1} = \frac{b}{1} \iff a \cdot 1 - b \cdot 1 = 0 \iff a = b$. We usually omit S and write j. We have that:

* $\forall s \in S$ the element $\frac{s}{1} \in S^{-1}A$ is invertible with inverse $\frac{1}{s}$.
* $S^{-1}A$ is a domain.
* $\operatorname{Frac} A = S^{-1}A$ with $S = A \setminus \{0\}$.

Proposition

A domain. $S \subseteq A$ multiplicative. Let B be a ring and $\varphi : A \to B$ ring hom s.t. $\forall s \in S$ $g(s)$ is a unit in B. Then, $\exists! \, \varphi' : S^{-1}A \to B$ s.t. $\varphi = \varphi' \cdot j$

$$\begin{tikzcd}
S^{-1}A \arrow[r, dashed, "\varphi'"] & B \\
A \arrow[u, "j"] \arrow[ur, "\varphi"'] \arrow[ur, phantom, shift left=3, "\circlearrowright"]
\end{tikzcd}$$

Proof

Existence: Not much choice really... Define $\varphi' : S^{-1}A \to B$ by $\varphi'\left(\frac{a}{s}\right) = \varphi(a)\varphi(s)^{-1}$, and hope it is well-defined ü But it clearly is: $\frac{a}{s} = \frac{a'}{s'} \iff as' = a's$

$$\Rightarrow \varphi(as') = \varphi(a's) \iff \varphi(a)\varphi(s') = \varphi(a')\varphi(s) \underset{\substack{\varphi(s),\,\varphi(s') \text{ units} \\ \text{of } B}}{\iff} \varphi(a)\varphi(s)^{-1} = \varphi(a')\varphi(s')^{-1}$$

$$\iff \varphi'\left(\frac{a}{s}\right) = \varphi'\left(\frac{a'}{s'}\right).$$

Clearly, $\varphi = \varphi' \circ j$:

$$\begin{array}{ccccc} a & \xmapsto{\ j\ } & \frac{a}{1} & \xmapsto{\ \varphi'\ } & \varphi(a)\varphi(1)^{-1} = \varphi(a) \\ \cap & & \cap & & \cap \\ A & & S^{-1}A & & B \end{array}$$

φ' can be easily verified to be a ring hom.

Uniqueness: Assume $\psi : S^{-1}A \to B$ is s.t. $\psi \circ j = \varphi$. Then, $\psi\left(\frac{a}{1}\right) = (\psi \circ j)(a) = \varphi(a)$ $\forall a \in A$.

Thus, $1 = \psi\left(\frac{s}{1} \cdot \frac{1}{s}\right) = \psi\left(\frac{s}{1}\right)\psi\left(\frac{1}{s}\right) = \varphi(s)\psi\left(\frac{1}{s}\right) \Rightarrow \psi\left(\frac{1}{s}\right) = \varphi(s)^{-1}$ in B $\forall s \in S \Rightarrow$

$$\psi\left(\frac{a}{s}\right) = \psi\left(\frac{a}{1}\right)\psi\left(\frac{1}{s}\right) = \varphi(a)\varphi(s)^{-1} = \varphi'\left(\frac{a}{s}\right).$$

$\blacksquare$

Remark:

Such a decomposition of every map $\varphi : A \to B$ with the mention property of B is called the <u>universal property</u> of rings of fractions.

Corollary

A domain, $K = \operatorname{Frac} A$. $S \subseteq A$ multiplicative. Then $S^{-1}A$ can be identified in a unique manner with the subring of A generated by $\{\frac{1}{s} \mid s \in S\}$.

Proof

Let $T = A \setminus \{0\}$. Then, $K = T^{-1}A$. Let $j_T : A \to K$, $j_S : A \to S^{-1}A$.

As $j_T(s)$ is invertible $\forall s \in S$, the previous proposition yields $\exists!\, \varphi : S^{-1}A \longrightarrow K$ s.t. $j_T = \varphi \circ j_S$. Further, $S^{-1}A$ being domain yields φ injective, and so $S^{-1}A$ can be identified in a unique way with its image under φ in K.

$$\begin{tikzcd} S^{-1}A \arrow[r, hook, "\varphi"] & K \\ A \arrow[u, "j_S"] \arrow[ur, "j_T"'] & \end{tikzcd}$$

This image is

$$\{\varphi(\tfrac{a}{s})\} = \{j_T(a)\, j_T(s)^{-1}\}$$

Proof of previous proposition

$$= \{\tfrac{a}{1} \cdot \tfrac{1}{s}\} = A[\{\tfrac{1}{s} \mid s \in S\}],$$

where we identify A with K's subring $j_T(A) = \{\frac{a}{1} \mid a \in A\}$. ∎

Ideals in Rings of Fractions

We'll typically study a ring of interest A by studying ring of fractions associated with A. In particular, we'd like to understand the relation between R's ideals and the ideals of $S^{-1}A$.

Lemma

A domain. $S \subseteq A$ multiplicative. Let I be an ideal of A. Let $S^{-1}I$ be the ideal of $S^{-1}A$ generated by $j_S(I)$. Then, $S^{-1}I = \{ \frac{i}{s} \mid i \in I, s \in S \}$.

Proof

An element of $S^{-1}I$ is of the form $\sum_{j=1}^{n} \frac{a_j}{s_j} \cdot \frac{i_j}{1}$. So, clearly $\frac{i}{s}$ with $i \in I$ $s \in S$ is $\in S^{-1}I$. By take common denominator and using that I is an ideal and S multiplicative, one can write $\sum_{j=1}^{n} \frac{a_j}{s_j} \cdot \frac{i_j}{1}$ as $\frac{i}{s}$ for $i \in I$, $s \in S$. ∎

Corollary

$S^{-1}I = S^{-1}A \iff I \cap S \neq \emptyset.$

Proof

$S^{-1}I = \{\frac{i}{s} \mid i \in I, s \in S\}$ by the previous lemma. If $S^{-1}I = S^{-1}A$ then $\exists s \in S, i \in I$ s.t. $\frac{i}{s} = \frac{1}{1} \Rightarrow i = s \Rightarrow I \cap S \neq \emptyset$.

On the other hand, if $0 \neq \alpha \in I \cap S$ then $\frac{1}{1} = \frac{\alpha}{\alpha} \in S^{-1}I \Rightarrow S^{-1}I = S^{-1}A$.

Claim

A domain, $S \subseteq A$ multiplicative. Let $j : A \hookrightarrow S^{-1}A$. Let J be an ideal of $S^{-1}A$, $I = j^{-1}(J)$. Then $S^{-1}I = J$.

Proof

$i \in I \Rightarrow \frac{i}{1} \in J$. $\forall s \in S$ $\frac{1}{s} \in S^{-1}A \Rightarrow \frac{i}{1}\frac{1}{s} \in J \Rightarrow S^{-1}I \subseteq J$. On the other hand, if $\frac{i}{s} \in J$ then $\frac{s}{1} \cdot \frac{i}{s} \in J \Rightarrow i \in J \Rightarrow i \in I \Rightarrow \frac{i}{s} \in S^{-1}I$.

Lemma

A domain. $S \subseteq A$ multiplicative set. Let $P \subseteq A$ be a prime ideal s.t. $P \cap S = \emptyset$.

Then $S^{-1}P$ is a prime ideal of $S^{-1}A$.

Proof

First, as we proved, since $P \cap S = \emptyset$, $S^{-1}P \neq S^{-1}A$. Now, let $\frac{a}{s}, \frac{b}{t} \in S^{-1}A$ with $\frac{a}{s} \cdot \frac{b}{t} \in S^{-1}P$. We proved that $\frac{a}{s} \cdot \frac{b}{t} = \frac{c}{r}$ for some $c \in P$ $r \in S$. Thus, $abr = stc \in P$. As $r \in S$ and $S \cap P = \emptyset$, one of $a, b \in P \Rightarrow$ one of $\frac{a}{s}, \frac{b}{t}$ $\in S^{-1}P$.

Proposition

A domain, $S \subseteq A$ mul subset. The map $j : A \hookrightarrow S^{-1}A$ induces a bijection

$$j^{-1} : \mathrm{Spec}(S^{-1}A) \longrightarrow \{ P \in \mathrm{Spec}(A) \mid P \cap S = \emptyset \}.$$

Proof

Given $Q \in \mathrm{Spec}(S^{-1}A)$ define $P = j^{-1}(Q)$. Note that $P \cap S = \emptyset$ as otherwise $Q = S^{-1}A$. Indeed, if $a \in P \cap S$ then $j(a) \in Q$ as $a \in P$ but, as we proved, $a \in S \Rightarrow j(a)$ is a unit of $S^{-1}A$.

<u>j^{-1} is injective</u>: if $Q_1, Q_2 \in \mathrm{Spec}(S^{-1}A)$ are such that $j^{-1}(Q_1) = j^{-1}(Q_2)$ then as proved, $S^{-1}(j^{-1}(Q_1)) = Q_1$, $S^{-1}(j^{-1}(Q_2)) = Q_2$ $\Rightarrow Q_1 = Q_2$.

j^{-1} is surjective:

Let $P \in \operatorname{Spec}(A)$, $P \cap S = \emptyset$. Clearly, $P \subseteq j^{-1}(S^{-1}P)$. Take $a \in j^{-1}(S^{-1}P)$. Then, $\exists\, p \in P$ $s \in S$ s.t. $j(a) = \frac{a}{1} = \frac{p}{s}$ $\Rightarrow$ $p = as$. As $s \in S$ and $S \cap P = \emptyset$, $a \in P$ $\Rightarrow$ $j^{-1}(S^{-1}P) \subseteq P$. ∎

Remark

The map j^{-1} and its inverse are inclusion-preserving.

Localization

Definition

A domain. $P \in \operatorname{Spec} A$. The ring $(A \setminus P)^{-1} A$, also denoted A_P is called the localization of A at P. The ideal $(A \setminus P)^{-1} P$ is then denoted PA_P.

Definition

A ring is a local ring if it has a unique maximal ideal.

Claim

A domain. $P \in \operatorname{Spec} A$. Then, $\operatorname{Spec}(A_P)$ is in inclusive-preserving bijection with the prime ideals of A contained in P. In particular, A_P is local with maximal ideal PA_P.

Proof

We proved an inclusive-preserving bijection

$Q \subseteq P$

$$\operatorname{Spec}(A_P) \rightleftarrows \{ Q \in \operatorname{Spec} A \mid Q \cap (A \setminus P) = \emptyset \}$$

The inverse of j^{-1} send $Q \in \operatorname{Spec} A$ $Q \subseteq P$ to Q_P. Thus $P_P = PA_P$ is $\in \operatorname{Spec}(A_P)$ and contains all prime ideals of A_P, let alone all maximal ideals of A_P. Hence, $\max A_P = \{PA_P\}$.

$\blacksquare$

<u>Claim</u>

A domain. $P \in \operatorname{Spec} A$. Then $ht(P) = \dim A_P$ and $\dim A = \sup\{\dim A_P \mid P \in \operatorname{Spec} A\}$.

<u>Proof</u>

The prime ideals in A contained in P are in inclusive-preserving bijection with the prime ideals of A_P. Thus, the chains of prime ideals in A_P ending at PA_P are in bijection with the chains of ideals in A ending at P.

Corollary

A domain. $\dim A = 1$. $S \subseteq A$ multiplicative s.t. $A \setminus S$ contains a maximal ideal P in A. Then, $\dim(S^{-1}A) = 1$. In particular, $\dim A_P = 1$.

Properties Inherited by Rings of Fractions

Proposition

A noetherian, $S \subseteq A$ multiplicative. Then, $S^{-1}A$ noetherian.

Proof

Take J ideal in $S^{-1}A$. Let $I = j^{-1}(J) = \alpha_1 A + \dots + \alpha_n A$. observe that

$J = j(\alpha_1) S^{-1}A + \dots + j(\alpha_n) S^{-1}A$. Indeed, we proved that $J = S^{-1}I = \{ \frac{i}{s} \mid i \in I, s \in S \}$.

But $i = \alpha_1 a_1 + \dots + \alpha_n a_n$. So, an element of J can be written as

$j(\alpha_1) \cdot \frac{a_1}{s} + \dots + j(\alpha_n) \cdot \frac{a_n}{s}$.

Proposition

A integrally closed domain. $S \subseteq A$ multiplicative. Then, $S^{-1}A$ is integrally closed domain.

Proof

Let $K = \operatorname{Frac} A$. Up to ring isomorphism, as proved, $K = \operatorname{Frac}(S^{-1}A)$. Let $0 \neq \alpha \in K$.

$\alpha = \frac{a}{b}$ $a, b \in A$ integral over $S^{-1}A$. Then $\exists\, f(y) \in (S^{-1}A)[y]$

$$f(y) = y^n + \frac{a_{n-1}}{s_{n-1}} y^{n-1} + \cdots + \frac{a_0}{s_0}$$

with $f(\alpha) = 0$. Let $s = s_0 \cdots s_{n-1}$. Then

$$0 = s^n f(\alpha) = \left(\frac{sa}{b}\right)^n + \frac{a_{n-1} \cdot s}{s_{n-1}} \left(\frac{sa}{b}\right)^{n-1} + \cdots + \frac{a_0}{s_0} \cdot s^n$$

$\Rightarrow \frac{sa}{b} \in K$ integral over A $\Rightarrow \frac{sa}{b} \in A$ $\Rightarrow \alpha = \frac{a}{b} \in S^{-1}A$.

A integrally closed

$\blacksquare$

Claim

A domain, $K = \text{Frac } A$. L/K finite field extension. Let B denote the i.c of A in L. $S \subseteq A$ multiplicative. Then $S^{-1}B$ is the integral closure of $S^{-1}A$ in L.

Proof

As B is an integral closure, B is integrally closed in its field of fractions L. Thus, by the previous claim, $S^{-1}B$ is integrally closed. We now show that every element $\frac{b}{s} \in S^{-1}B$ is integral over $S^{-1}A$. Let $f(y) = y^n + a_{n-1}y^{n-1} + \cdots + a_0 \in A[y]$ be s.t $f(b) = 0$. Let $g(y) = y^n + \frac{a_{n-1}}{s} y^{n-1} + \cdots + \frac{a_0}{s^n} \in (S^{-1}A)[y]$. Then $g\left(\frac{b}{s}\right) = f(b) = 0$ and so $\frac{b}{s}$ integral over $S^{-1}A$.

Corollary

A D.D. $S \subseteq A$ multiplicative s.t. $A \setminus S$ contains a nonzero prime ideal of A.

Then, $S^{-1}A$ is a D.D.

Proof

We proved that $S^{-1}B$ is noetherian and i.c. As $A \setminus S \supseteq P \neq 0$ prime ideal and since $\dim A = 1$, P maximal, it follows by what we proved that $\dim S^{-1}A = 1$.

∎