I ✓

Plan.

* Spectral graph theory 101
* Expander graphs → Zig Zag Product

Reingold Vadhan Wigderson 2000

"Free" union of perfect matchings

Marcus Spielman Srivastava ... 2015

Fundamental questions we can't answer.

A detour to Free Probability Theory

$\exists$ ONE more probability theory!

Voiculescu 1985

* Freeness $\approx$ independence
* Central limit theorem
* Analytic machinery

II

You are here

Free Probability

101

Plan

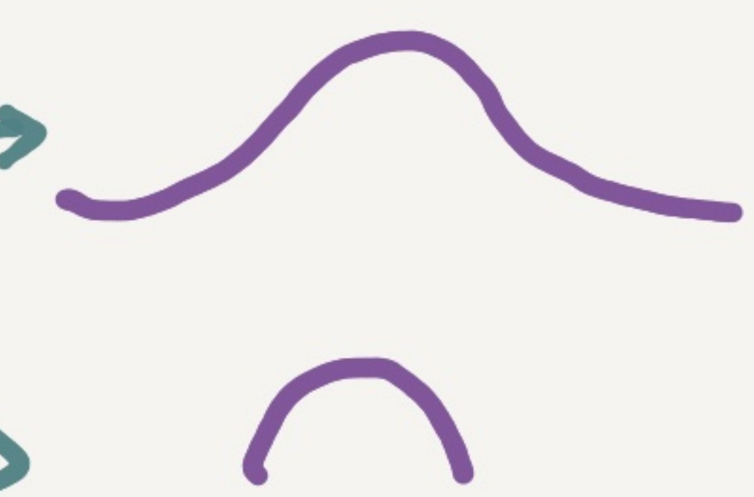
* Non-commutative probability spaces

  abstracting / algebraizing (nc) probability spaces

* Freeness

  The "new" independence

* Central limit theorem



* Analytic perspective & $\boxplus$

* MSS with a cheat.

Classical independence is an assumption on the random variables that gives a way of determining mixed moments by the marginal moments.

$$\mathbb{E}[abab] = \mathbb{E}[a^2]\,\mathbb{E}[b^2]$$

Freeness is THE ONLY other way for doing that!

Def. A non-commutative probability space (ncps) is

ring & $\mathbb{C}$-vector space

$$(\mathcal{A}, \varphi)$$

unital $\mathbb{C}$-algebra

unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$

The elements $a \in \mathcal{A}$ are called (nc) random variables.

We think of $\varphi(a)$ as "$\mathbb{E}[a]$" and assume φ is tracial: $\varphi(ab) = \varphi(ba)$.

Example.

A = $n \times n$ matrices

φ = normalized trace

$$\varphi(a) = \frac{1}{n}\sum_{i=1}^{n} a_{ii} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i(a)$$

Expectation of sampling $\lambda \sim \mathrm{Spec}(a)$

Example. Let G be a group, and consider the group algebra

$$\mathbb{C}G \triangleq \left\{ \sum_{g\in G} \alpha_g g \;\middle|\; \underset{\substack{\cap \\ \mathbb{C}}}{\alpha_g} = 0 \text{ almost always} \right\}$$

$$\varphi : \mathbb{C}G \longrightarrow \mathbb{C}$$

$$\sum_g \alpha_g g \longmapsto \alpha_e$$

identity in G

We typically equip our ncps with a $*$-operation

$$* : \mathcal{A} \to \mathcal{A}$$
$$a \mapsto a^*$$

s.t.

- $(a^*)^* = a$
- $(ab)^* = b^* a^*$
- $(a+b)^* = a^* + b^*$
- $\varphi(a^* a) \geq 0$ with equality iff $a=0$

With this we can define $a \in \mathcal{A}$ to be

- self adjoint $\quad a = a^*$
- unitary $\quad a^* a = a a^* = 1$
- normal $\quad a^* a = a a^*$

and prove useful results such as

$$|\varphi(b^* a)|^2 \leq \varphi(a^* a)\varphi(b^* b)$$

Cauchy-Schwarz

Example.

$\mathcal{A}$ = $n \times n$ real matrices

φ = normalized trace

$$\varphi(a) = \frac{1}{n} \sum_{i=1}^{n} a_{ii} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i(a)$$

$$a^* = a^*$$ usual matrix conjugation

Example. Let G be a group, and consider the group algebra

$$\mathbb{C}G \triangleq \left\{ \sum_{g \in G} \alpha_g g \;\middle|\; \alpha_g = 0 \text{ almost always} \right\}$$

$$\varphi : \mathbb{C}G \longrightarrow \mathbb{C}$$

$$\sum \alpha_g g \mapsto \alpha_e$$

identity in G

$$g^* = g^{-1}$$

Analytic Distributions

$aa^* = a^*a$

Let $(\mathcal{A}, \varphi)$ be a ncps, $a \in \mathcal{A}$ normal. If $\exists$ compactly supported probability measure μ on $\mathbb{C}$ s.t.

$$\int z^k \bar{z}^l \, d\mu(z) = \varphi(a^k (a^*)^l)$$

for all $k, l \in \mathbb{N}$, then μ is uniquely determined and is called the analytic distribution of a.

Haar Unitary.

$u^*u = uu^* = 1$

Let $(\mathcal{A}, \varphi)$ be a ncps. $u \in \mathcal{A}$ is **Haar unitary** if it is unitary &

$$\varphi(u^k) = \begin{cases} 1 & k = 0 \\ 0 & \text{o.w.} \end{cases}$$

The corresponding analytic distribution is the uniform distribution over the unit circle in $\mathbb{C}$ (exercise).

Freeness

Def. Let $(\mathcal{A}, \varphi)$ be a ncps.
$a, b \in \mathcal{A}$ are free if all centered, alternating mixed moments vanish.

E.g.

$a-\varphi(a)$

$$\varphi(\overline{a}\,\overline{b}) = 0$$

$b^3-\varphi(b^3)$

$$\varphi(\overline{a^2}\,\overline{b}\,\overline{a}\,\overline{b^3}) = 0$$

$$\vdots$$

Let's explore.

$$0 = \varphi((a-\varphi(a))(b-\varphi(b)))$$

$$= \varphi(ab - a\varphi(b) - \varphi(a)b + \varphi(a)\varphi(b))$$

$$= \varphi(ab) - \varphi(a)\varphi(b)$$

$$\Longrightarrow \quad \varphi(ab) = \varphi(a)\varphi(b)$$

as in the classical case.

But...

$$\varphi(\bar{a}\bar{b}\bar{a}\bar{b}) = 0 \quad \Rightarrow$$

$$\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2$$

compared to classical ind

$$\varphi(abab) = \varphi(a^2)\varphi(b^2)$$

Why don't we encounter freeness in our daily lives?

* Commuting rv are free only when one of them is constant.
* Freeness in an "∞-dim phenomena."

Classical ind & freeness are the only two "consistent" ways of determining mixed moments from marginals.

Fun exercise.

Let $(\mathcal{A}, \varphi)$ be a ncps. Let $a, b, u \in \mathcal{A}$ such that u is Haar unitary that is free from $\{a, b\}$.

Prove that a & ubu^* are free.

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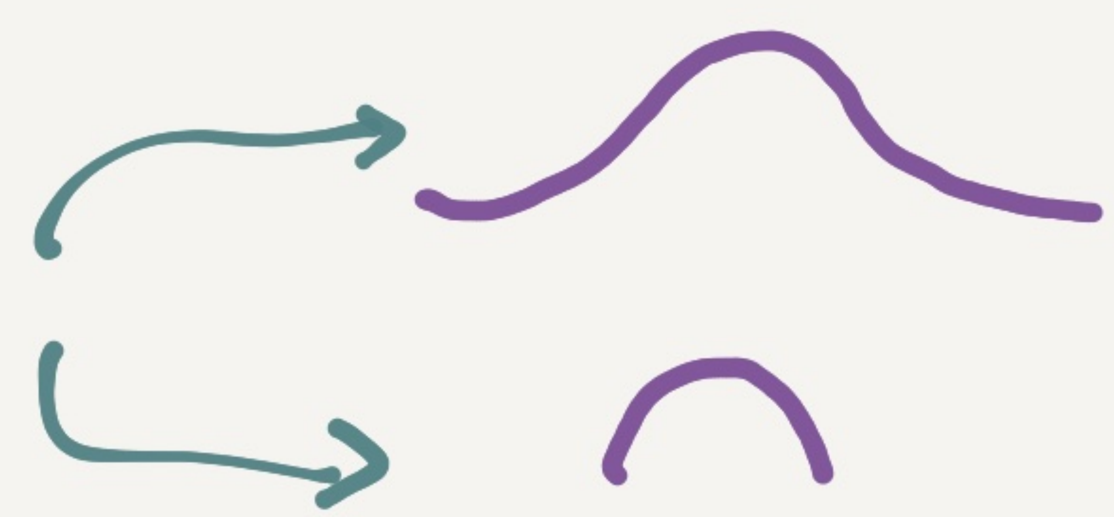
abstracting / algebraizing (nc) probability spaces

✓ Freeness ← The "new" independence

* Central limit theorem



* Analytic perspective &
* MSS with a cheat.

A CLT asks about the behavior of

$$\lim_{N \to \infty} \frac{a_1 + \cdots + a_N}{\sqrt{N}}$$

$\varphi(a_i) = 0$
$\varphi(a_i^2) = 1$

where convergence is in terms of moments:

$$a_N \longrightarrow a \quad \text{means} \quad \forall m \quad \lim_{N \to \infty} \varphi_N(a_N^m) = \varphi(a^m)$$

Fix $m, N \geq 1$

$$\varphi((a_1 + \cdots + a_N)^m) = \sum_{r:[m]\to[N]} \varphi(a_{r(1)} \cdots a_{r(m)})$$

Note that, say,

$$\varphi(a_1 a_2 a_2 a_3 a_1 a_2) = \varphi(a_9 a_1 a_1 a_5 a_9 a_1)$$

Generally, the value of $\varphi(a_{r(1)} \cdots a_{r(m)})$ depends on r only through the information on which of the indices are equal.

We encode this information by a partition

$$\pi = \{V_1, \dots, V_s\} \text{ of } [n]$$

where

$$\forall i,j \quad r(i) = r(j) \iff \exists \ell \quad i,j \in V_\ell$$

E.g.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ a_1 & a_2 & a_2 & a_3 & a_1 & a_2 \\ a_9 & a_1 & a_1 & a_5 & a_9 & a_1 \end{array}$$

correspond to $\{\{1,5\}, \{2,3,6\}, \{4\}\}$

With this

#r-s that are associated with π

$$\varphi((a_1 + \cdots + a_N)^m) = \sum_{\substack{\pi \text{ partition} \\ \text{of } [m]}} k_\pi A^N_\pi$$

only this depends on N

The common value of $\varphi(a_{r(1)} \cdots a_{r(m)})$ for all r associated with π

Observation. π containing a singleton does not contribute:

Both in classical & free ind

$$k_\pi = \varphi(\sim a_r \sim) = \underbrace{\varphi(a_r)}_{=0}\,\varphi(\sim\ \sim)$$

In particular, we can restrict to π-s with $|\pi| \le n/2$.

Now,

$$A_{\pi}^{N} = N(N-1)\cdots(N-|\pi|+1) \xrightarrow[N\to\infty]{} N^{|\pi|}$$

$$\Longrightarrow \lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\cdots+a_N}{\sqrt{N}}\right)^m\right) =$$

$$\lim_{N\to\infty} \sum_{\pi} K_{\pi} \frac{A_{\pi}^{N}}{N^{N/2}} =$$

$$\sum_{\pi} K_{\pi} \lim_{N\to\infty} N^{|\pi|-m/2}$$

So the only surviving π-s are pairings.

$$\lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\dots+a_N}{\sqrt{N}}\right)^m\right) = \sum_{\substack{\pi \text{ pairing} \\ \text{of } [m]}} k_\pi$$

CLASSICAL CLT:

$\forall \pi$ pairing $\quad k_\pi = \varphi(a_i^2)\varphi(a_j^2)\dots = 1$

$\Rightarrow$ LHS = # pairings of $[m]$ = $(m-1)(m-3)\cdots$

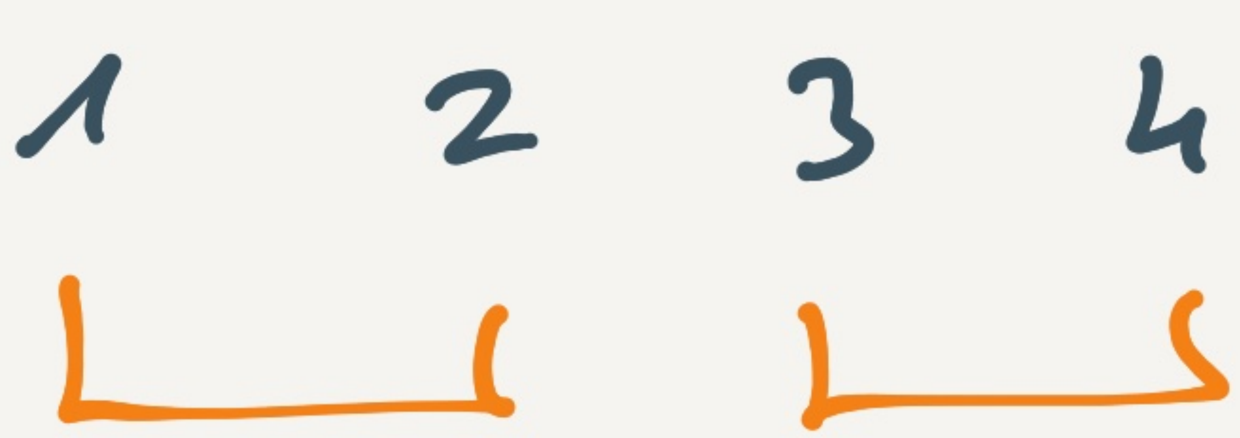
Exercise.

$$\frac{1}{\sqrt{2\pi}} \int t^m e^{-t^2/2}\, dt = \begin{cases} (m-1)(m-3)\cdots & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

Classical CLT $\blacksquare$

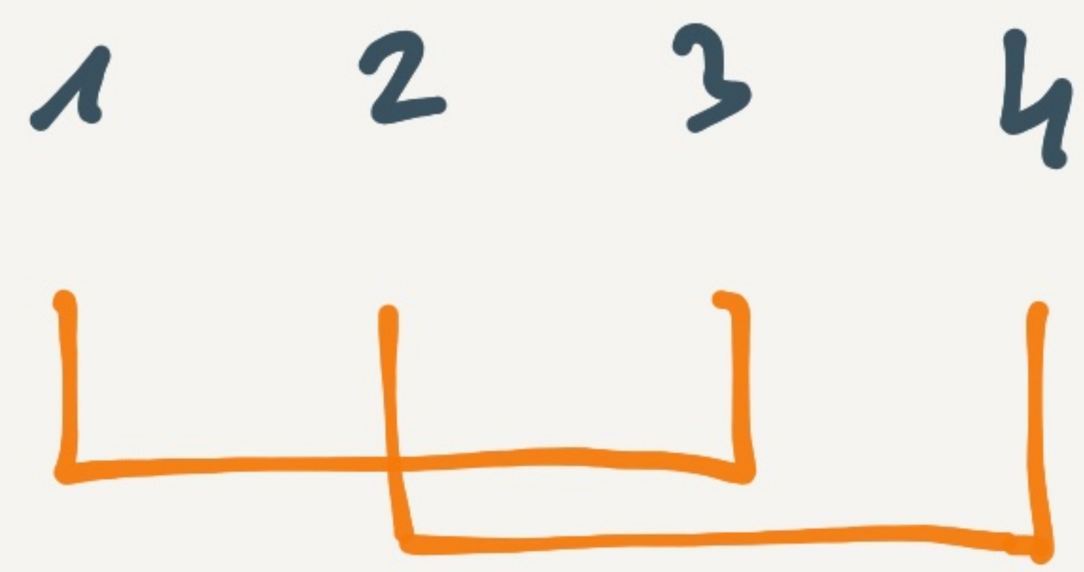
Free CLT. Lets look at some pairings:

* $\pi = \{\{1,2\},\{3,4\}\}$



$$\varphi(a^2 b^2) = \varphi(a^2)\varphi(b^2)$$
$$= 1$$

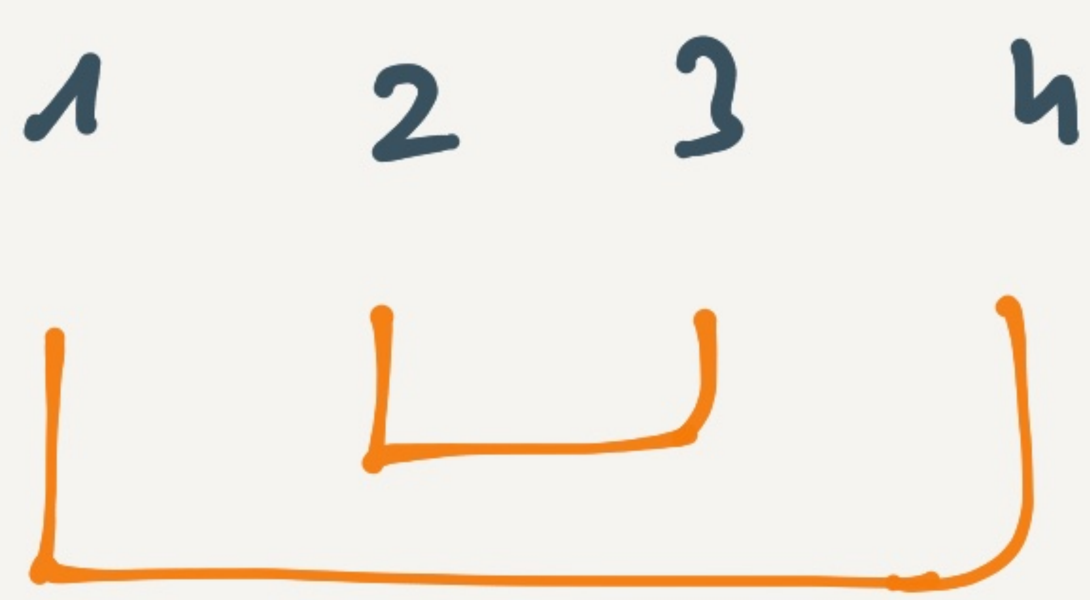
*



$$\varphi(abab) = 0$$

a, b are free & centered

*

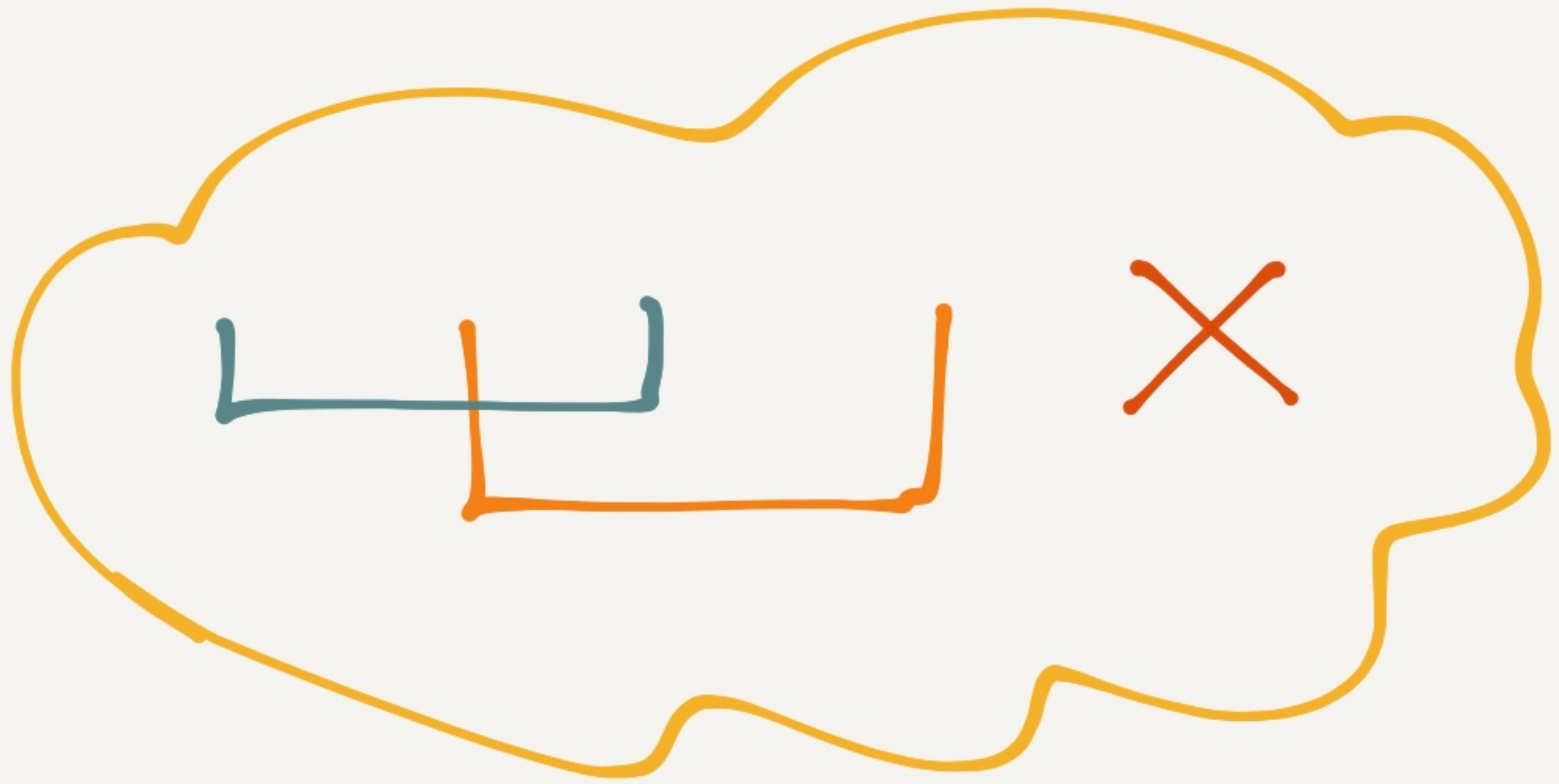


$$\varphi(ab^2a) = \varphi(a^2)\varphi(b^2) = 1$$

Generally, the surviving pairing correspond to non-crossing pairings.

A pairing π is non-crossing if $\forall \{p_1 < p_2\}, \{q_1 < q_2\} \in \pi$ it does not hold that

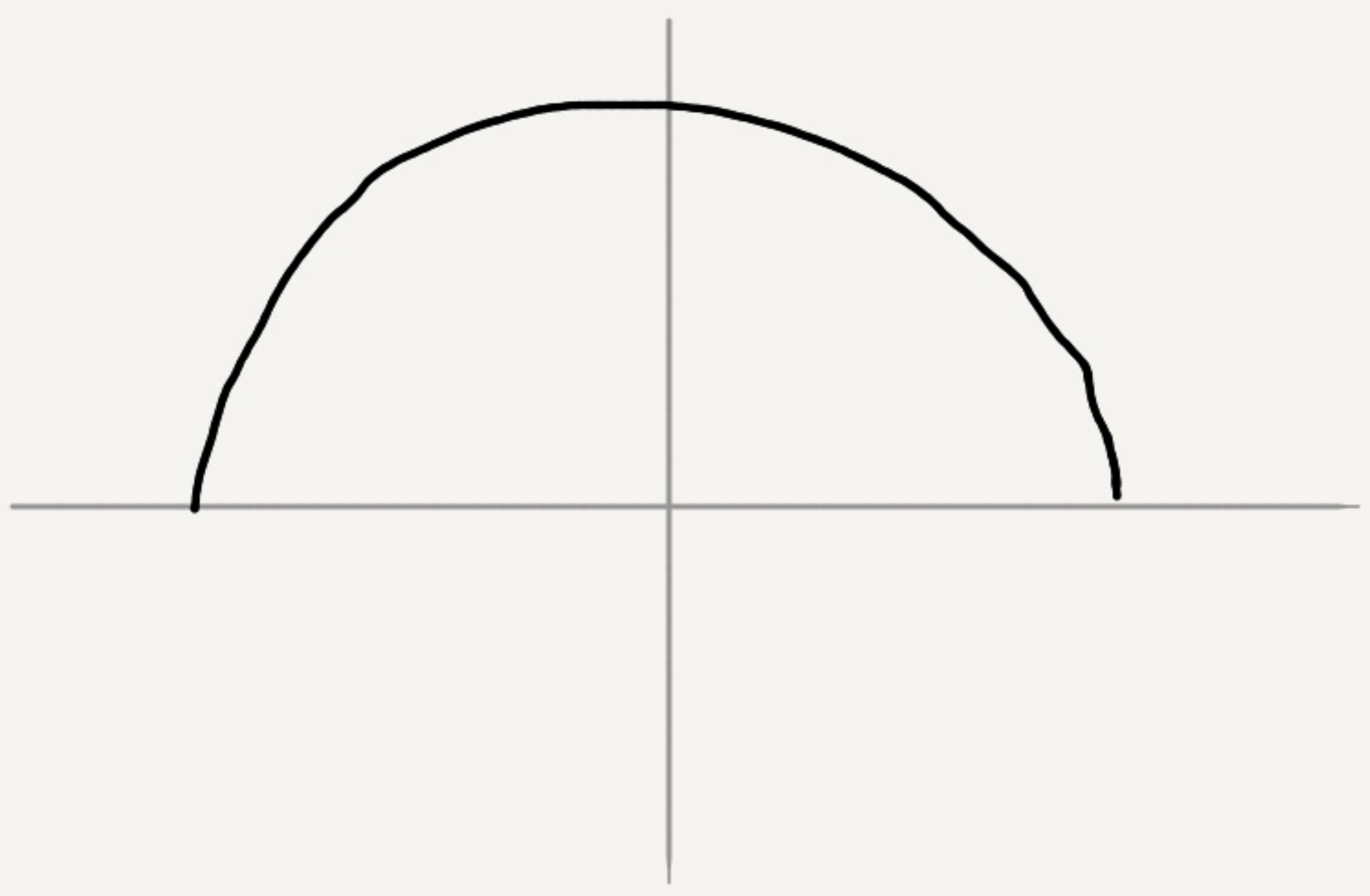
$$p_1 < q_1 < p_2 < q_2$$



Thm. The number of non-crossing pairings of $[2m]$ is C_m - the m-th Catalan number.

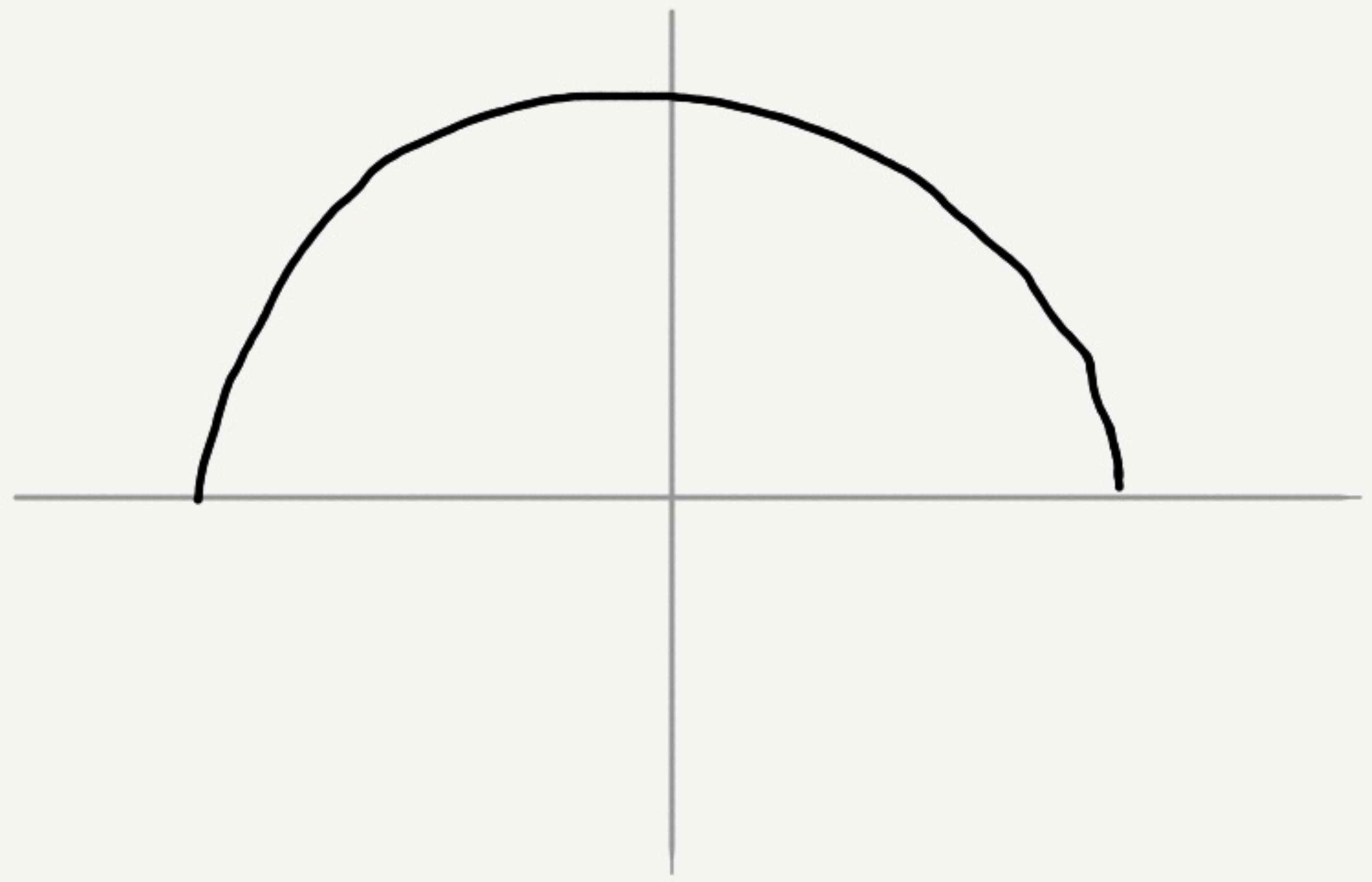
The analytic distribution whose (even) moments are the Catalan numbers is the semicircular distribution

$$\frac{1}{2\pi}\sqrt{4-x^2}$$



Free probability theory

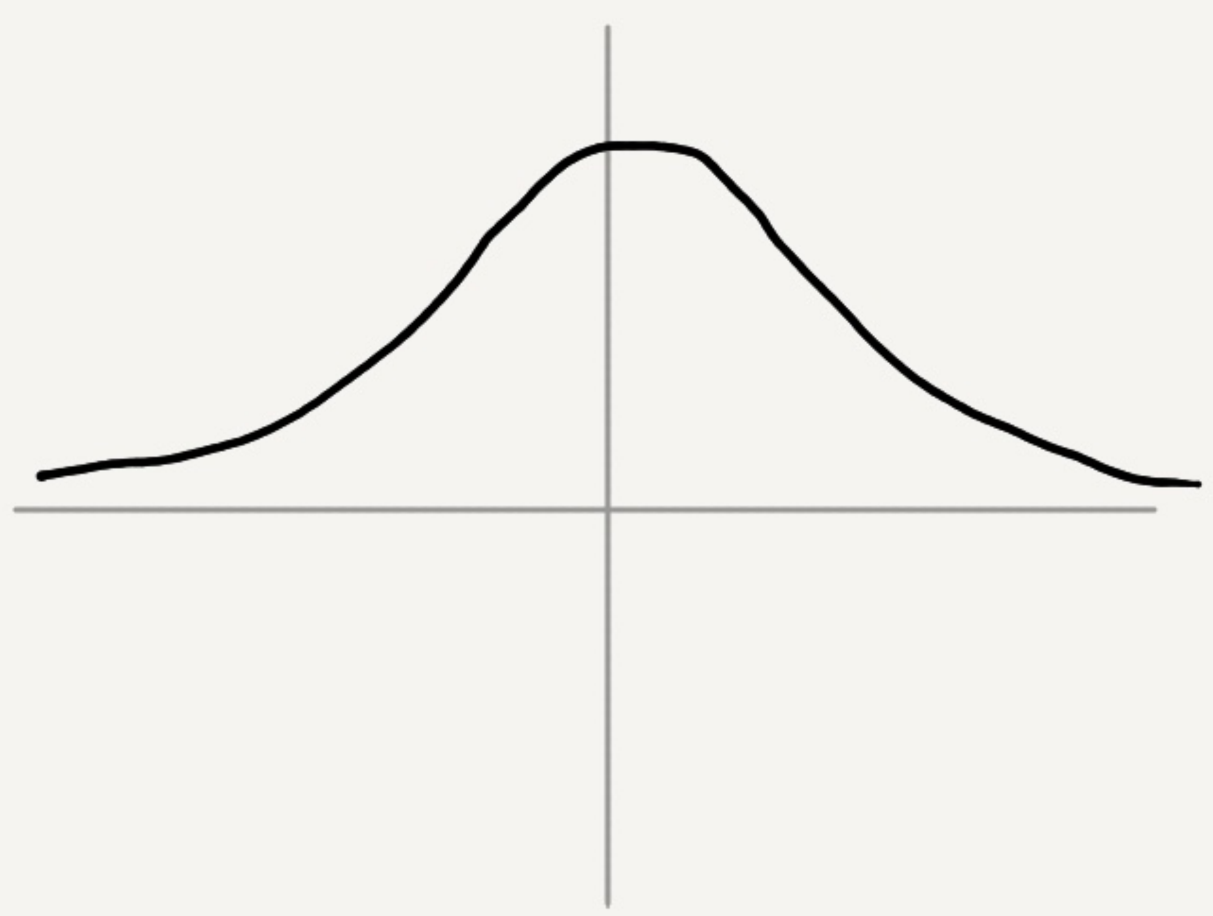
$$\frac{1}{2\pi}\sqrt{4-x^2}$$



non-crossing partitions

Classical probability theory

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



partitions

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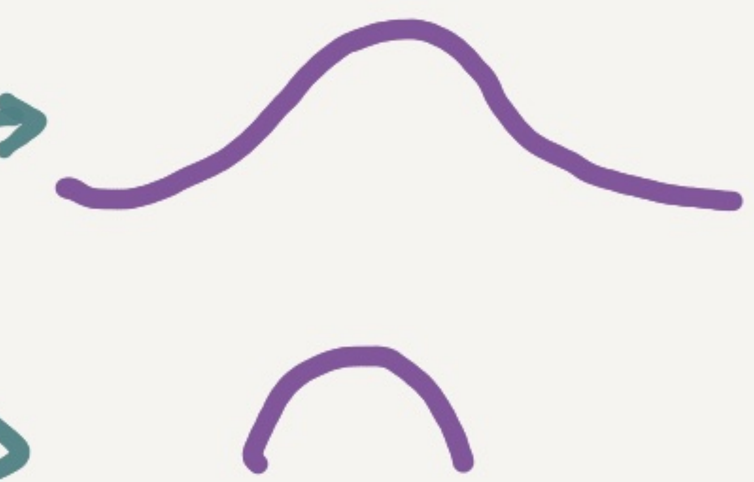
✓ Freeness ← The "new" independence

✓ Central limit theorem



⇒ * Analytic perspective & $\boxplus$

* MSS with a cheat.

The Cauchy Transform

Def. For a distribution μ define the Cauchy Transform

$$G_\mu(z) = \int \frac{d\mu(t)}{z-t} = \frac{1}{z}\int \sum_{n=0}^{\infty}\left(\frac{t}{z}\right)^n d\mu(t)$$

Analytic on $\mathbb{C}^+ \to \mathbb{C}^-$

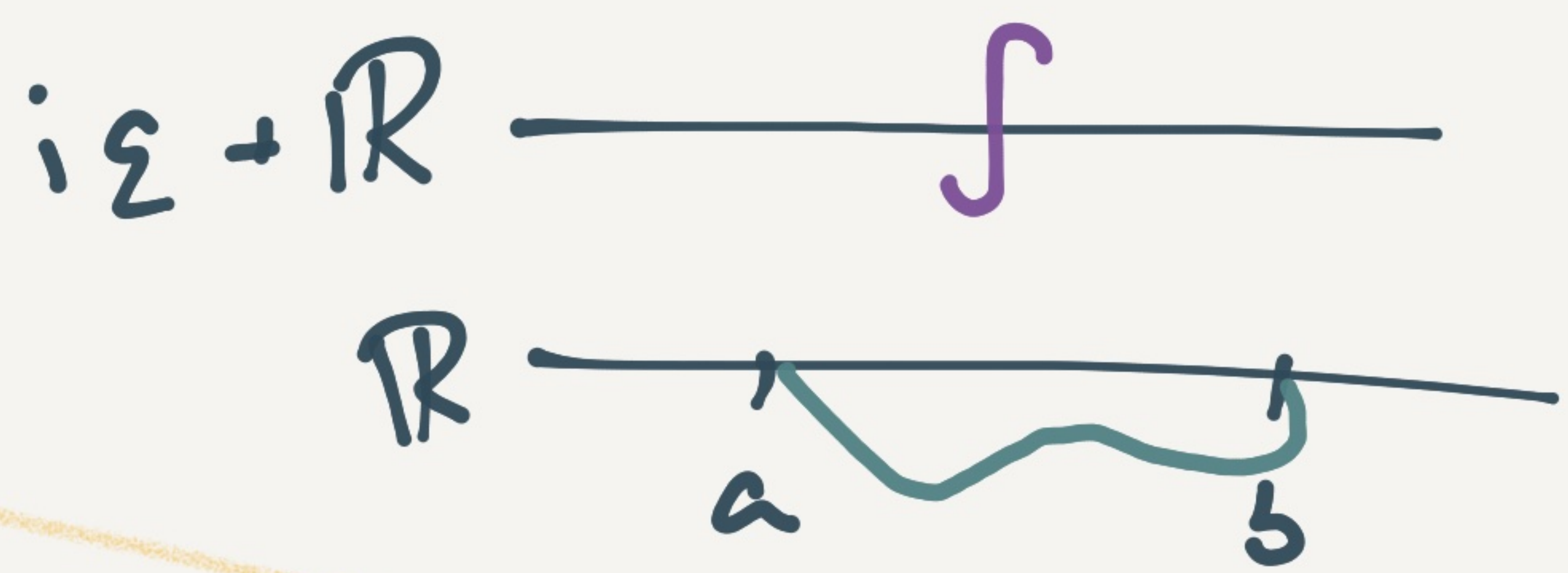
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \underbrace{\int t^n d\mu(t)}_{m_n(\mu) = \varphi(a^n)}$$

$\Rightarrow$ The moments of μ are encoded as the coefficients of $G_\mu(z)$ when expanding around ∞.

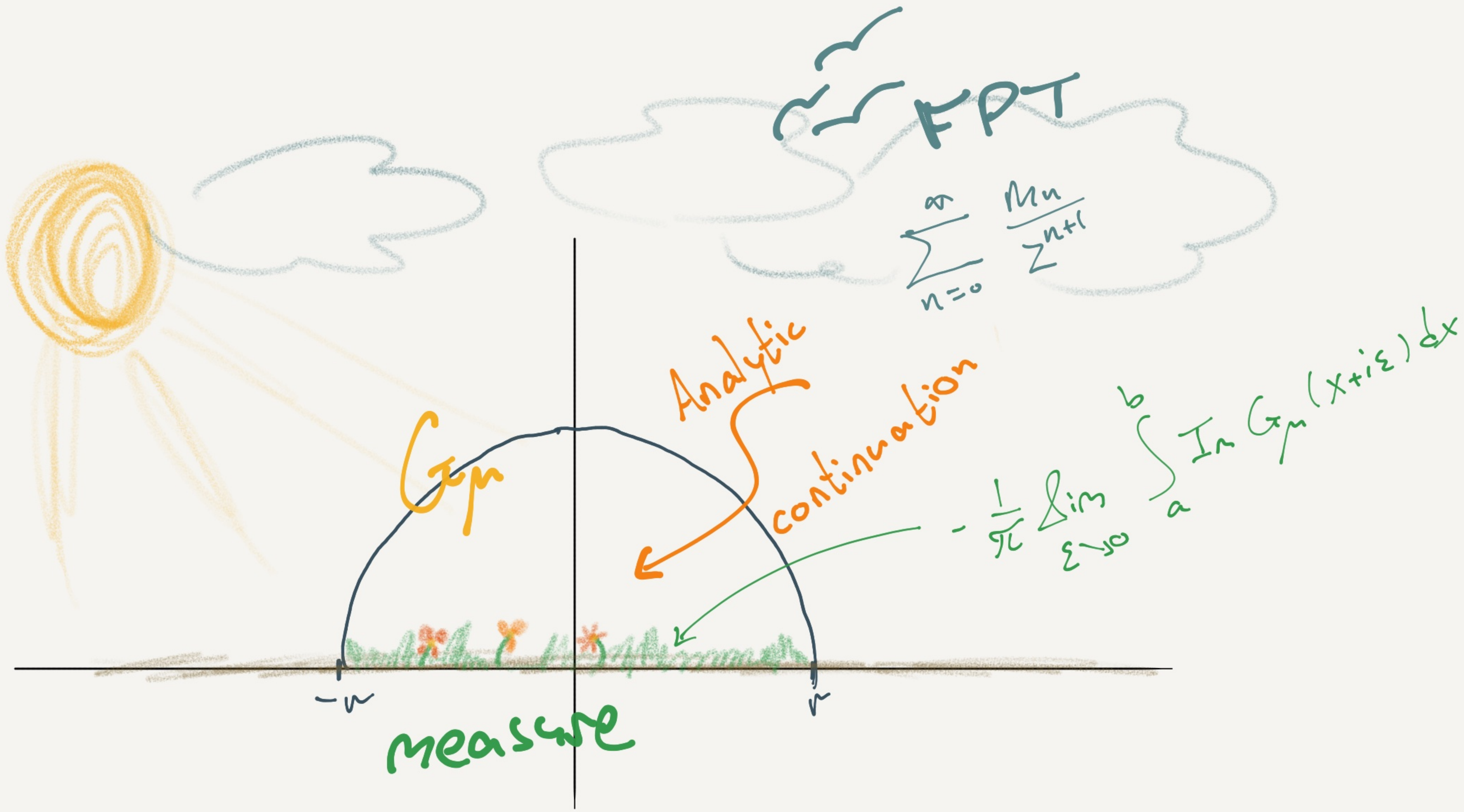
The Cauchy transform also encodes in a nice way the probability measure:

Theorem (Stieltjes Inversion Formula).

$$\mu((a,b)) + \tfrac{1}{2}\mu(\{a,b\}) = -\frac{1}{\pi}\lim_{\varepsilon \searrow 0} \int_a^b \operatorname{Im} G_\mu(x+i\varepsilon)\,dx$$



Heaven & Earth Theorem



The Additive Free Convolution

Given distributions μ, ν we define

$$\mu \boxplus \nu$$

The additive free convolution

as follows:

I Construct a ncps $(\mathcal{A}, \varphi)$ with two free random variables a, b whose marginals encode μ, ν:

$$\varphi(a^k) = \int z^k \, d\mu(z)$$

$$\varphi(b^k) = \int z^k \, d\nu(z)$$

II Define $\mu \boxplus \nu$ as the distribution induced by the moments of $a+b$

$$\varphi\left((a+b)^k\right) = \int z^k \, d(\mu \boxplus \nu)(z)$$

free

$$\begin{tikzcd}
\text{Algebra} & & \text{Measures} \\
a \arrow[dd, bend right] & & \mu \arrow[ll, squiggly] \\
b \arrow[d] & & \nu \arrow[ll, squiggly] \\
a+b \arrow[rr, squiggly] & & \mu \boxplus \nu
\end{tikzcd}$$

The inverse under composition of G_μ is denoted by K_μ:

$$G_\mu(K_\mu(z)) = z$$

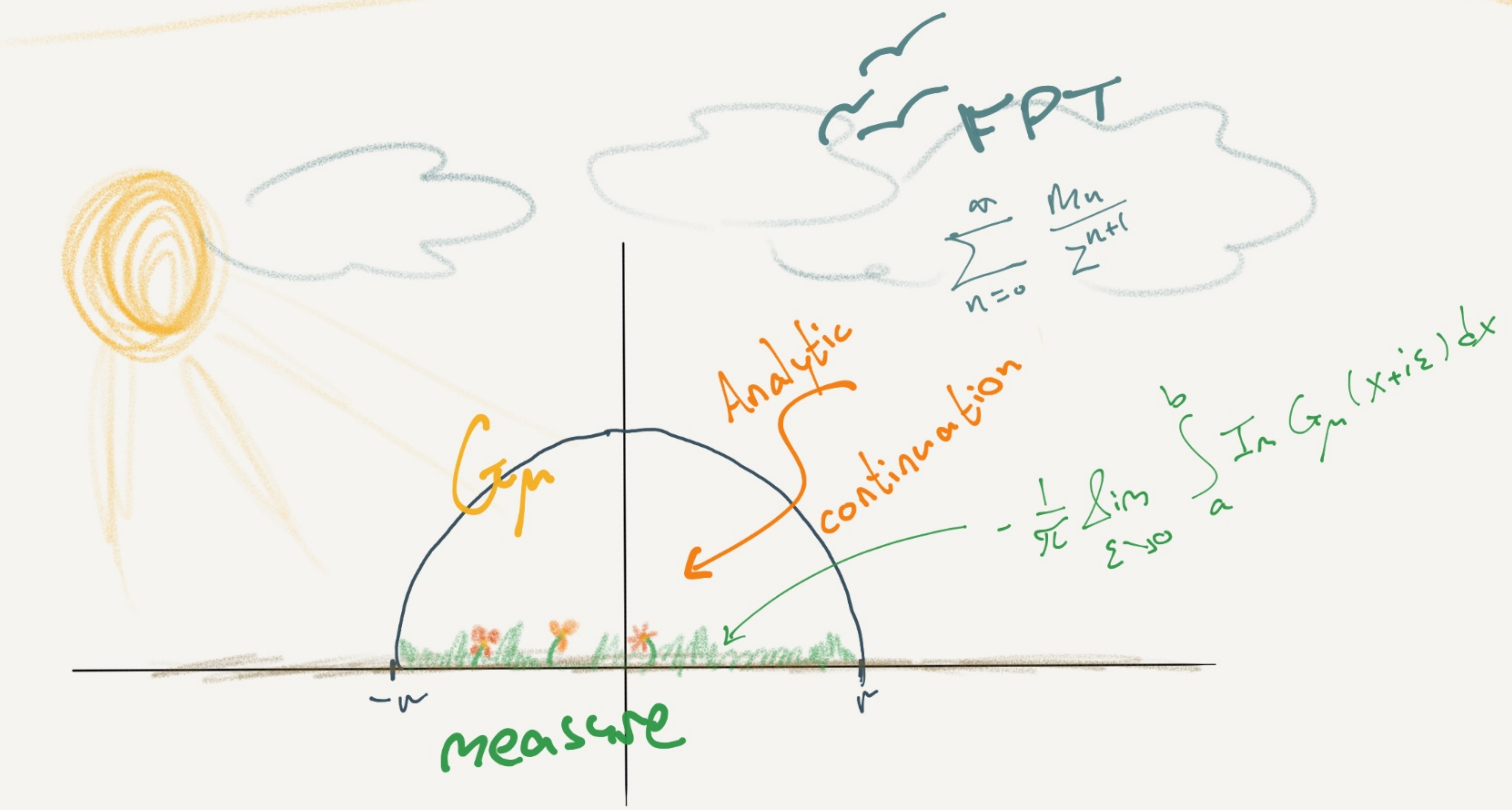
The K-transform Theorem.

$$K_{\mu \boxplus \nu}(z) = K_\mu(z) + K_\nu(z) - \frac{1}{z}$$

Recap

Def. For a distribution μ define the **Cauchy Transform**

$$G_\mu(z) = \int \frac{d\mu(t)}{z-t} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \underbrace{\int t^n \, d\mu(t)}_{m_n(\mu) = \varphi(a^n)}$$



Heaven & Earth Theorem

$$\begin{tikzcd}
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a \arrow[dd, bend right] & & \mu \arrow[ll, squiggly] \\
b \arrow[d] & & \nu \arrow[ll, squiggly] \\
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\end{tikzcd}$$

free

The additive free convolution

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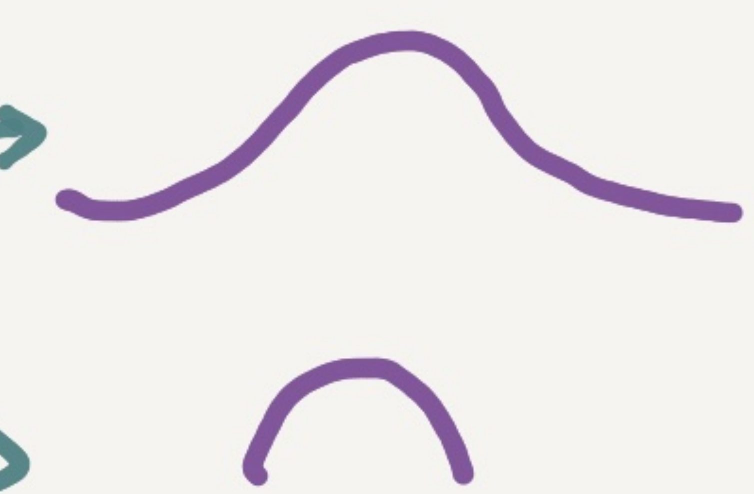
- ✓ Freeness

  The "new" independence

- ✓ Central limit theorem

- ✓ Analytic perspective & $\boxplus$

- * MSS with a cheat.

Let's get back to our question:

$$\lambda_2\left(P_1 \mu P_1^T + P_2 \mu P_2^T + \cdots + P_d \mu P_d^T\right) = ?$$

$\circlearrowleft$ $\mathrm{Spec} = \begin{cases} +1 & n/2 \\ -1 & n/2 \end{cases}$

If the "freeness philoshopy" is any good

$$\mu_d \triangleq \underbrace{\mu \boxplus \cdots \boxplus \mu}_{d \text{ copies}} \qquad \mu = \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$$

should shed some light.

$$\mu = \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$$

$$G_\mu(z) = \int \frac{d\mu(t)}{z-t} = \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right) = \frac{z}{z^2-1}$$

$$\underset{\text{invert}}{\Longrightarrow} \qquad K_\mu(z)^2 - \frac{1}{z}K_\mu(z) - 1 = 0$$

$$\Longrightarrow \qquad K_\mu(z) = \frac{1 \pm \sqrt{1+4z^2}}{2z}$$

Hence,

K-Transform Thm

$$K_{\mu_d}(z) = d\,K_{\mu}(z) - \frac{d-1}{z} = \frac{d\sqrt{1+4z^2} - d + 2}{2z}$$

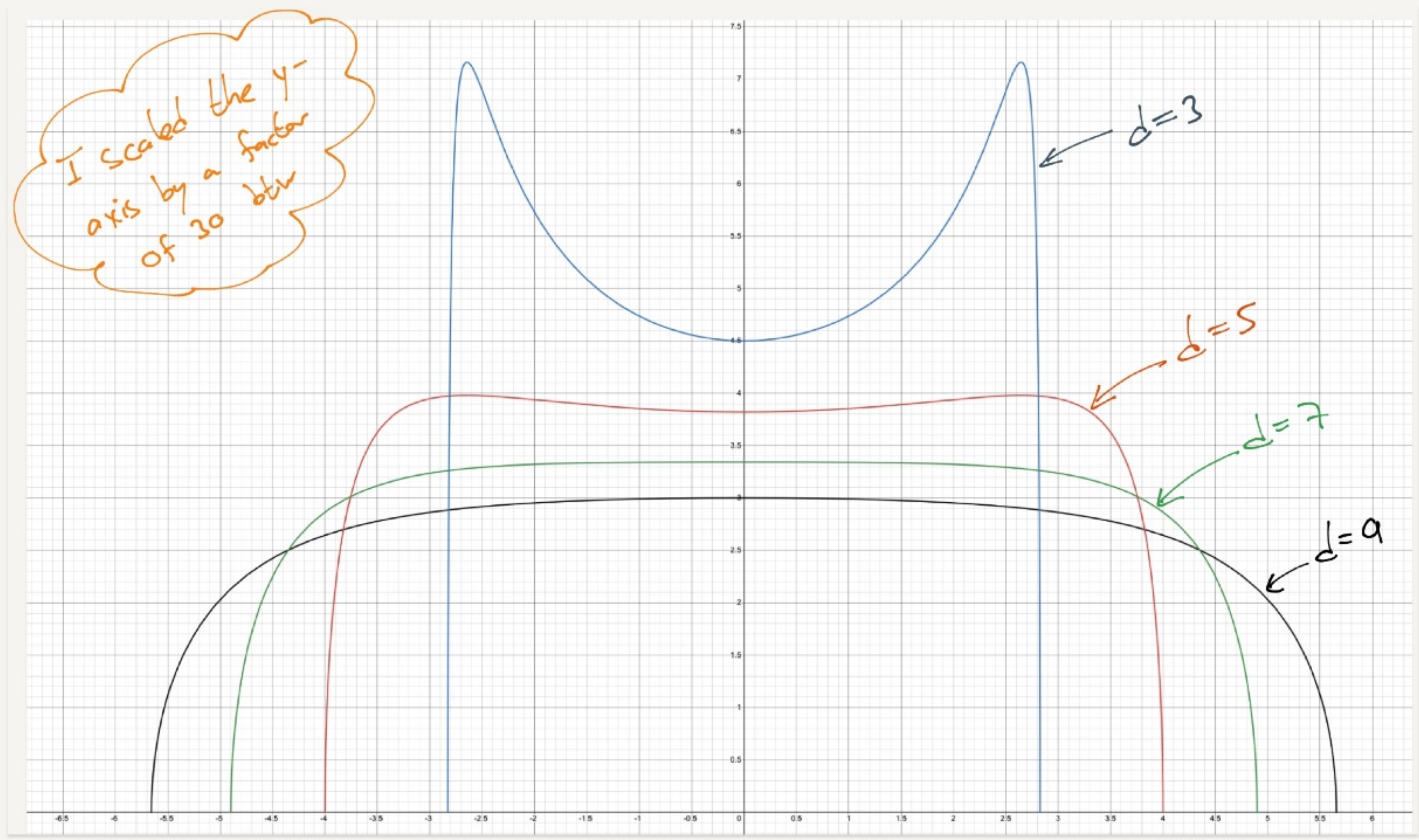
$\Longrightarrow$ invert again

$$G_{\mu_d}(z) = \frac{(2-d)z + d\sqrt{z^2 - 4(d-1)}}{2(z^2 - d^2)}$$

$\Longrightarrow$ Stieltjes inversion formula

$$\mu_d(t) = \begin{cases} \dfrac{d\sqrt{4(d-1) - t^2}}{2\pi(d^2 - t^2)} & |t| < \underline{\underline{2\sqrt{d-1}}} \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_d(t) = \begin{cases} \dfrac{d\sqrt{4(d-1)-t^2}}{2\pi(d^2-t^2)} & |t| < 2\sqrt{d-1} \\ 0 & \text{o.w.} \end{cases}$$



This is the Kesten-McKay distribution - the "limit spectrum" of d-regular graphs.

As $d \to \infty$ $\longrightarrow$ semicircular!

We got the correct result in this model of our problem that we studied, but what does this say about our original problem on graphs union?

MSS proof is based on a variant of free probability they develop called finite free probability.

Plan.

* Spectral graph theory 101
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Reingold Vadhan Wigderson 2000

→ "Free" union of perfect matchings

Marcus Spielman Srivastava ... 2015

Fundamental questions we can't answer.

A detour to Free Probability Theory

I ✓

∃ ONE more probability theory! Voiculescu 1985

* Freeness ≈ independence
* Central limit theorem
* Analytic machinery

II ✓

Finite FPT (interlacing, quadrature, ...)

* One-sided Ramanujan graphs
* Zig Zag revisited

III