Recap

Gil Cohen

January 22, 2025

The Different

Definition 1

Let F/L be an extension of E/K with F/E finite and separable. Let $\mathfrak{p}$ be a prime divisor of E/K with valuation ring $\mathcal{O}_\mathfrak{p}$ and integral closure $\mathcal{O}'_\mathfrak{p}$ in F. Let

$$C_\mathfrak{p} = t_\mathfrak{p}\mathcal{O}'_\mathfrak{p}$$

be the complementary module over $\mathcal{O}_\mathfrak{p}$.

We define the different exponent of $\mathfrak{P}/\mathfrak{p}$ by

$$d(\mathfrak{P}/\mathfrak{p}) = -v_\mathfrak{P}(t_\mathfrak{p}).$$

The different of F/E if defined by

$$\mathrm{Diff}(F/E) = \sum_{\mathfrak{p}\in\mathbb{P}(E)} \sum_{\mathfrak{P}/\mathfrak{p}} d(\mathfrak{P}/\mathfrak{p})\mathfrak{P}.$$

Dedekind Different Theorem

Theorem 2 (Dedekind Different Theorem)

Let F/L *be a finite separable extension of* E/K. *Let* $\mathfrak{p} \in \mathbb{P}(E)$ *and* $\mathfrak{P} \in \mathbb{P}(F)$ *lying over* $\mathfrak{p}$. *Then,*

1. $d(\mathfrak{P}/\mathfrak{p}) \geq e(\mathfrak{P}/\mathfrak{p}) - 1$; *and*
2. $d(\mathfrak{P}/\mathfrak{p}) = e(\mathfrak{P}/\mathfrak{p}) - 1 \quad \iff \quad \mathrm{char}\, K \nmid e(\mathfrak{P}/\mathfrak{p})$.

Corollary 3

With the above notations,

$$d(\mathfrak{P}/\mathfrak{p}) = 0 \quad \iff \quad e(\mathfrak{P}/\mathfrak{p}) = 1$$

In particular, for almost all $\mathfrak{p}, \mathfrak{P}/\mathfrak{p}$ *we have that* $e(\mathfrak{P}/\mathfrak{p}) = 1$.

Hurwitz Genus Formula

Theorem 4

Let F/L *be a finite separable extension of* E/K*. Let* $g_{\mathsf{E}}, g_{\mathsf{F}}$ *be the corresponding genera. Then,*

$$2g_{\mathsf{F}} - 2 = \frac{[\mathsf{F} : \mathsf{E}]}{[\mathsf{L} : \mathsf{K}]} \cdot (2g_{\mathsf{E}} - 2) + \deg \mathrm{Diff}(\mathsf{F}/\mathsf{E}).$$

A lemma about the dual basis

We have the following lemma about dual bases.

Lemma 5

Let F/L *be a degree* n *separable extension of* E/K *s.t.*

$$\mathrm{F} = \mathrm{E}(y) \qquad y \in \mathrm{F}.$$

Let $\varphi(T) \in \mathrm{E}[T]$ *be the minimal polynomial of* y *over* E, *and write*

$$\varphi(T) = (T - y)(c_0 + c_1 T + c_2 T^2 + \cdots + c_{n-1} T^{n-1}),$$

with $c_i \in \mathrm{F}$. *Then, the dual basis of* $1, y, y^2, \ldots, y^{n-1}$ *is given by*

$$\frac{c_0}{\varphi'(y)}, \ldots, \frac{c_{n-1}}{\varphi'(y)}.$$

A bound on the different exponent

Theorem 6

Let F/L *be a finite separable extension of* E/K *s.t.*

$$\mathsf{F} = \mathsf{E}(y) \qquad y \in \mathsf{F}.$$

Let $\mathfrak{p} \in \mathbb{P}(\mathsf{E})$ *be s.t.* $y \in \mathcal{O}'_{\mathfrak{p}}$.

Let $\varphi(T) \in \mathcal{O}_{\mathfrak{p}}[T]$ *be the minimal polynomial of* y *over* E.

Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_r \in \mathbb{P}(\mathsf{F})$ *be the prime divisors lying over* $\mathfrak{p}$. *Then,*

$$\forall i \in [r] \quad d(\mathfrak{P}_i/\mathfrak{p}) \leq \upsilon_{\mathfrak{P}_i}(\varphi'(y)).$$

The different exponent and local bases

Theorem 7

Let F/L *be a finite separable extension of* E/K *s.t.*

$$\mathsf{F} = \mathsf{E}(y) \qquad y \in \mathsf{F}.$$

Let $\mathfrak{p} \in \mathbb{P}(\mathsf{E})$ *be s.t.* $y \in \mathcal{O}'_{\mathfrak{p}}$.

Let $\varphi(T) \in \mathcal{O}_{\mathfrak{p}}[T]$ *be the minimal polynomial of* y *over* E.

Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_r \in \mathbb{P}(\mathsf{F})$ *be the prime divisors lying over* $\mathfrak{p}$. *Then,*

$$\mathcal{O}'_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}[y] \quad \Longleftrightarrow \quad \forall i \in [r] \quad d(\mathfrak{P}_i/\mathfrak{p}) = v_{\mathfrak{P}_i}(\varphi'(y)).$$

The different exponent and local bases

Corollary 8

Let F/L *be a finite separable extension of* E/K *s.t.*

$$\mathsf{F} = \mathsf{E}(y) \qquad y \in \mathsf{F}.$$

Let $\mathfrak{p} \in \mathbb{P}(\mathsf{E})$ *be s.t.* $y \in \mathcal{O}'_{\mathfrak{p}}$.

Let $\varphi(T) \in \mathcal{O}_{\mathfrak{p}}[T]$ *be the minimal polynomial of* y *over* E.

Assume that

$$\forall \mathfrak{P}/\mathfrak{p} \qquad v_{\mathfrak{P}}(\varphi'(y)) = 0.$$

Then, $\mathfrak{p}$ *is unramified in* F/E *and* $\mathcal{O}_{\mathfrak{p}}[y] = \mathcal{O}'_{\mathfrak{p}}$.