Mobius Inversion

Based on Nica-Speicher Chapter 10

Example. Suppose $f, g : \mathbb{N} \to \mathbb{C}$ are s.t.

$$g(x) = \sum_{y \le x} f(y)$$

May want to compare with the continuous analog: $g(x) = \int_{-\infty}^{x} f(y)\,dy$

underlying poset

o—o—o—o ...

$\mu = 1 \;\; -1 \;\; 0 \;\; 0 \;\cdots$

Express f in terms of g ...

$$f(x) = g(x) - g(x-1)$$

$f(x) = g'(x)$

Example. $f, g : \mathbb{N} \to \mathbb{C}$ s.t. $f(n) = \sum_{d \mid n} g(d)$. Then,

$$g(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) f(d)$$

underlying poset $(\mathbb{N}, \mid)$, with the famous μ

where if $n = p_1^{a_1} \cdots p_k^{a_k}$ then

The famous Mobius inversion formula

$$\mu(n) = \begin{cases} (-1)^k & a_1 = \cdots = a_k = 1 \\ 0 & \text{o.w.} \end{cases}$$

Closer to home. Recall from the CLT proof,

$$\lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\cdots+a_N}{\sqrt{N}}\right)^2\right) = \sum_{\pi \text{ pairings}} \kappa_\pi$$

moments free cumulants

we'll later write $\varphi(a_1\cdots a_n) = \sum_{\pi\in NC(n)} \kappa_\pi[a_1,\dots,a_n]$

and we'll need the machinery for inverting this - writing cumulants in terms of moments. We'll need to understand the Mobius inversion formula for $NC(n)$.

Interestingly, to this end we'll need to consider bivariate functions.

Incidence Algebras

Def. A poset $(P, \leq)$ is locally finite if

$$\forall x, y \in P \quad [x,y] \text{ is finite.}$$

Def. Let $(P, \leq)$ be a locally finite poset. The incidence algebra of $(P, \leq)$ is the $\mathbb{C}$-vector space

under point-wise addition

$$A(P) = \{ f : P \times P \to \mathbb{C} \mid f(x,y) \neq 0 \Rightarrow x \leq y \}.$$

As for multiplication, given $f, g \in A(P)$ define $f * g \in A(P)$ by

$$(f * g)(x,y) = \sum_{z \in P} f(x,z)\, g(z,y)$$

really only $z \in [x,y]$ contribute to the sum so P being a locally finite poset makes this well-defined

The function $\delta \in A(P)$

$$\delta(x,y) = \begin{cases} 1 & x=y \\ 0 & \text{o.w} \end{cases}$$

is a multiplicative unit: $f = \delta * f = f * \delta$.

It can be verified that $A(P)$ is associative. Hence, $A(P)$ is a unital associative algebra.

$A(P)$ in matrices.

For a finite poset, an element $f \in A(P)$ can be encoded as a $|P| \times |P|$ matrix where the order of rows & cols respect $\leq$ by $F_{ij} = f(i,j)$.

if row i appears before row j then $i \leq j$

convience yourself this is always possible

Note that F is an upper triangular matrix. Moreover,

$$(FG)_{ij} = \sum_k F_{ik} G_{kj} = \sum_k f_{ik} g_{kj} = (f * g)(i,j)$$

Thus, $A(P) \hookrightarrow$ upper triangular matrices.

as a $\mathbb{C}$-algebra

This gives a calculation-free way of proving associativity

Examples.

Chain

* $(P = [n], \leq)$. Then,

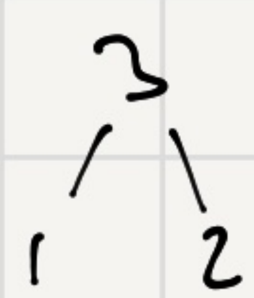
$$A(P) \cong \text{upper triangular matrices}$$

As a $\mathbb{C}$-algebra

* $P = [n]$ with an antichain partial order. $(x \leq y \iff x = y)$

$$A(P) \cong \text{diagonal matrices.}$$

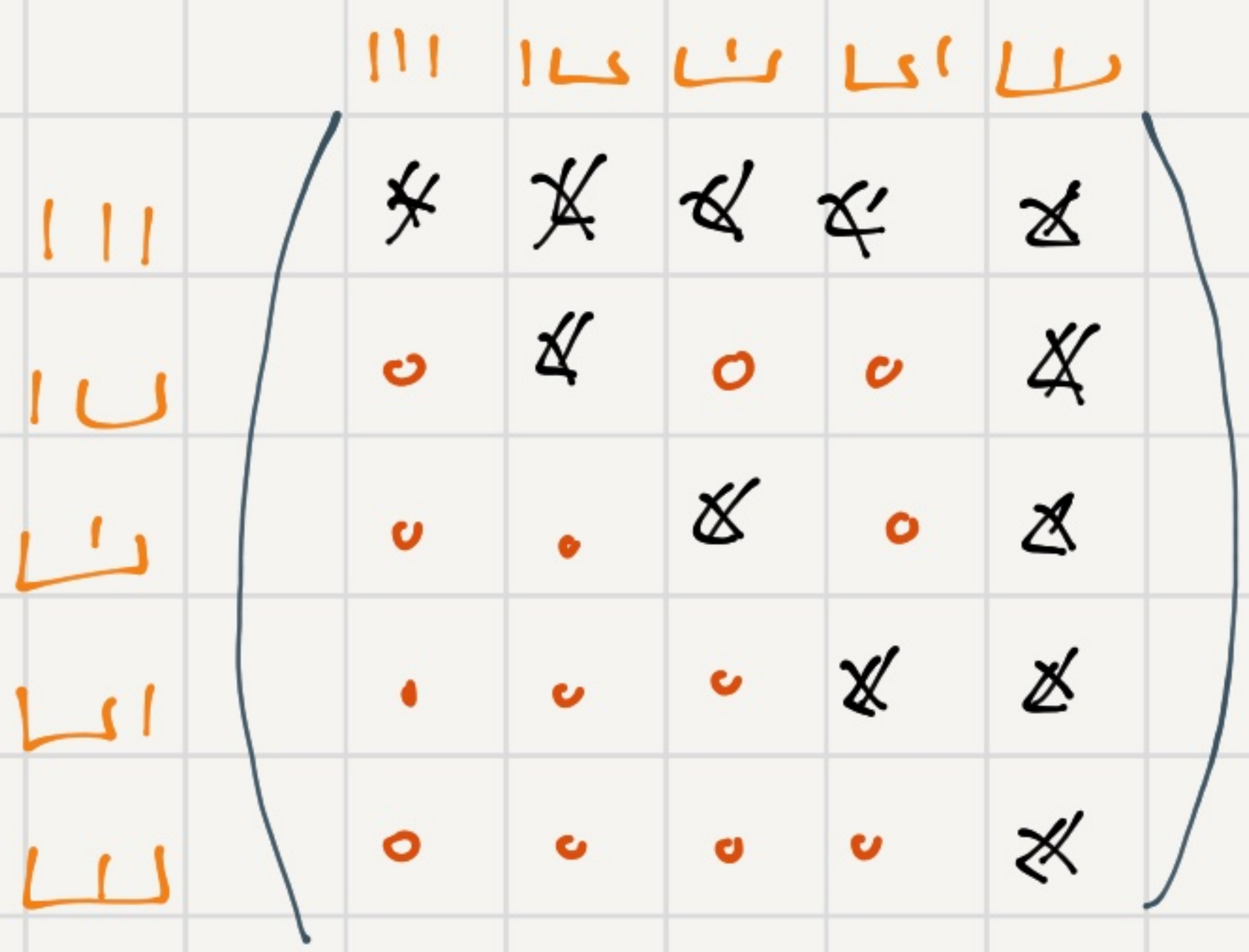
$\mathbb{C}$-alg

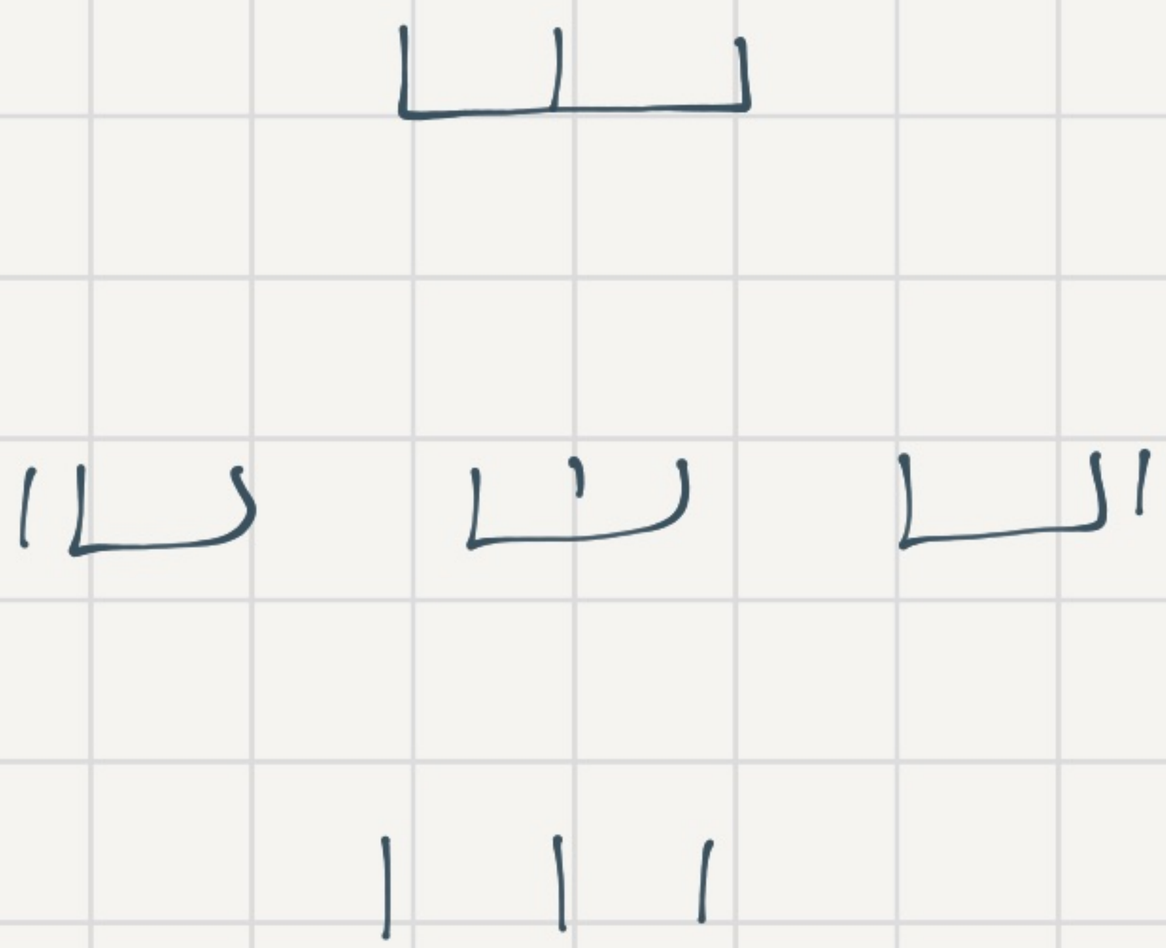
* $P =$ 

$$f \mapsto \begin{array}{cccc} & 1 & 2 & 3 \\ 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \hspace{-7em} \begin{array}{c} \\ \begin{pmatrix} f_{11} & 0 & f_{13} \\ 0 & f_{22} & f_{23} \\ 0 & 0 & f_{33} \end{pmatrix} \end{array}$$

$$A(P) \cong \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

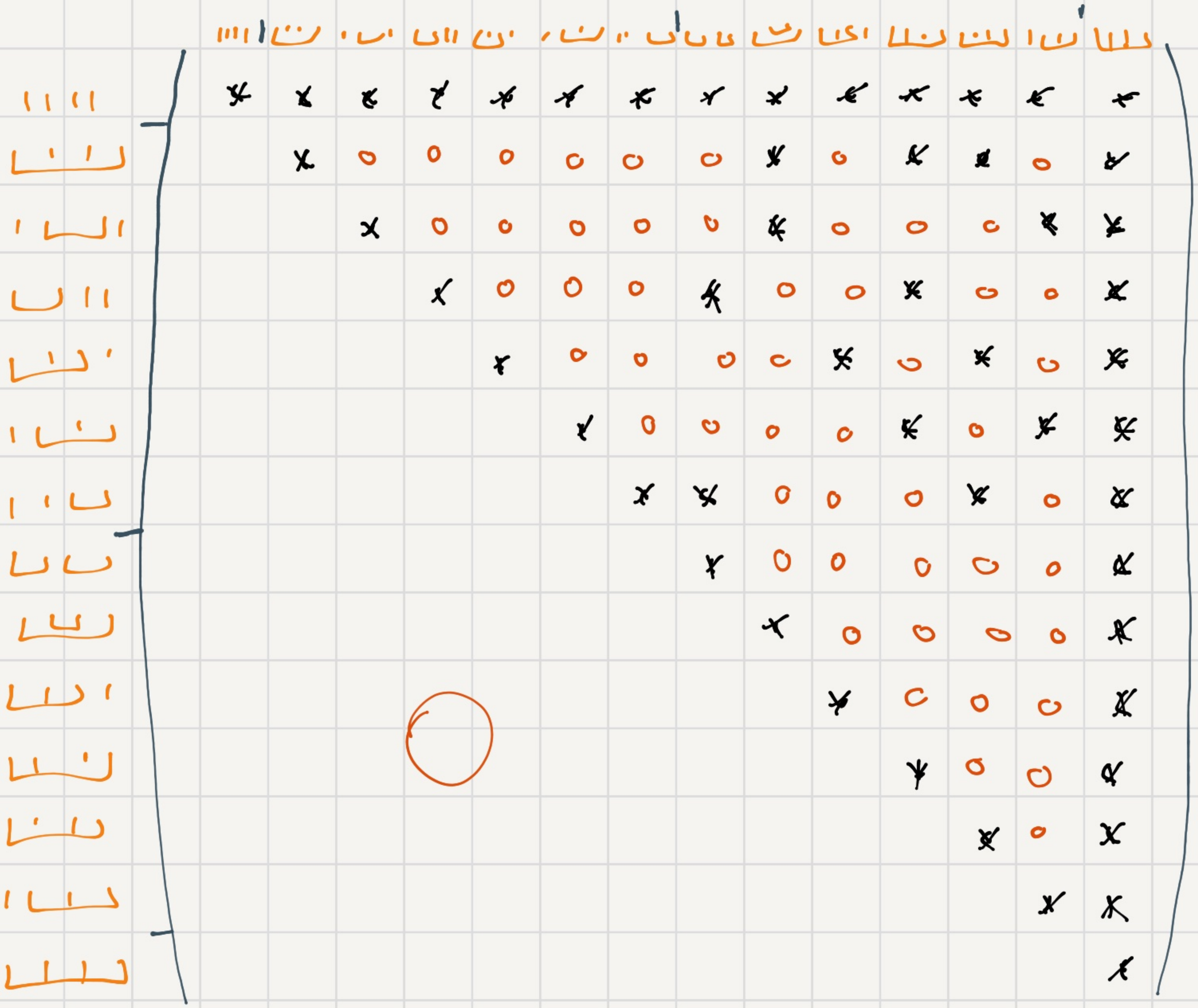
NC(3)





$\cancel{*}$ $NC(4)$

I'm regretting this already ☺



Inversion in $A(P)$

Note. If $f * h = \delta$ & $h * g = \delta$ then

$$f = f * \delta = f * (h * g) = (f * h) * g = \delta * g = g$$

so left & right inverse, if exist, are equal in a unital associative algebra.

locally finite poset

Theorem. Let $f \in A(P)$. Then f is invertible $\iff$ $f(x,x) \neq 0$ $\forall x \in P$.

Before pf. The proof must have something to do with the poset structure. E.g.

$$\begin{pmatrix} 1 & \underset{=}{1} & 0 \\ 0 & 1 & \underset{=}{1} \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

non zero

not a poset

pf. $\Rightarrow$ obvious. We prove $\Leftarrow$ by induction on the interval's $[x,y]$ length, defined as the largest $\ell \geq 0$ s.t.

$$x = p_0 < p_1 < \cdots < p_\ell = y$$

where $p_0, \dots, p_\ell \in P$.

Note that length $[x,y] = 0$ $\iff$ $x = y$

length 0

Base case. $x=y$. If g is s.t. $f * g = \delta$ then

$$1 = \delta(x,x) = (f * g)(x,x) = f(x,x)\, g(x,x)$$

$$\Longrightarrow \quad g(x,x) = f(x,x)^{-1}$$

recall $f(x,x) \neq 0$

Induction step. Let $x<y$. Then,

induction

$$0 = \delta(x,y) = (f * g)(x,y) = \sum_{x \le z \le y} f(x,z)\, g(z,y) = f(x,x)\, g(x,y) + \sum_{x < z \le y} f(x,z)\, g(z,y).$$

$$\Longrightarrow \quad g(x,y) = -f(x,x)^{-1} \cdot \sum_{x < z \le y} f(x,z)\, g(z,y).$$

also using $f(x,x) \neq 0$

We proved f has a right-inverse but by the above remark it is also the left inverse

The Mobius function

Def. For a locally finite poset $(P, \leq)$ define

Note $\zeta(x,x) = 1 \neq 0$ $\forall x$

$$\zeta(x,y) = \begin{cases} 1 & x \leq y \\ 0 & \text{o.w.} \end{cases}$$

The inverse of ζ, denoted as μ, is called the Mobius function of P.

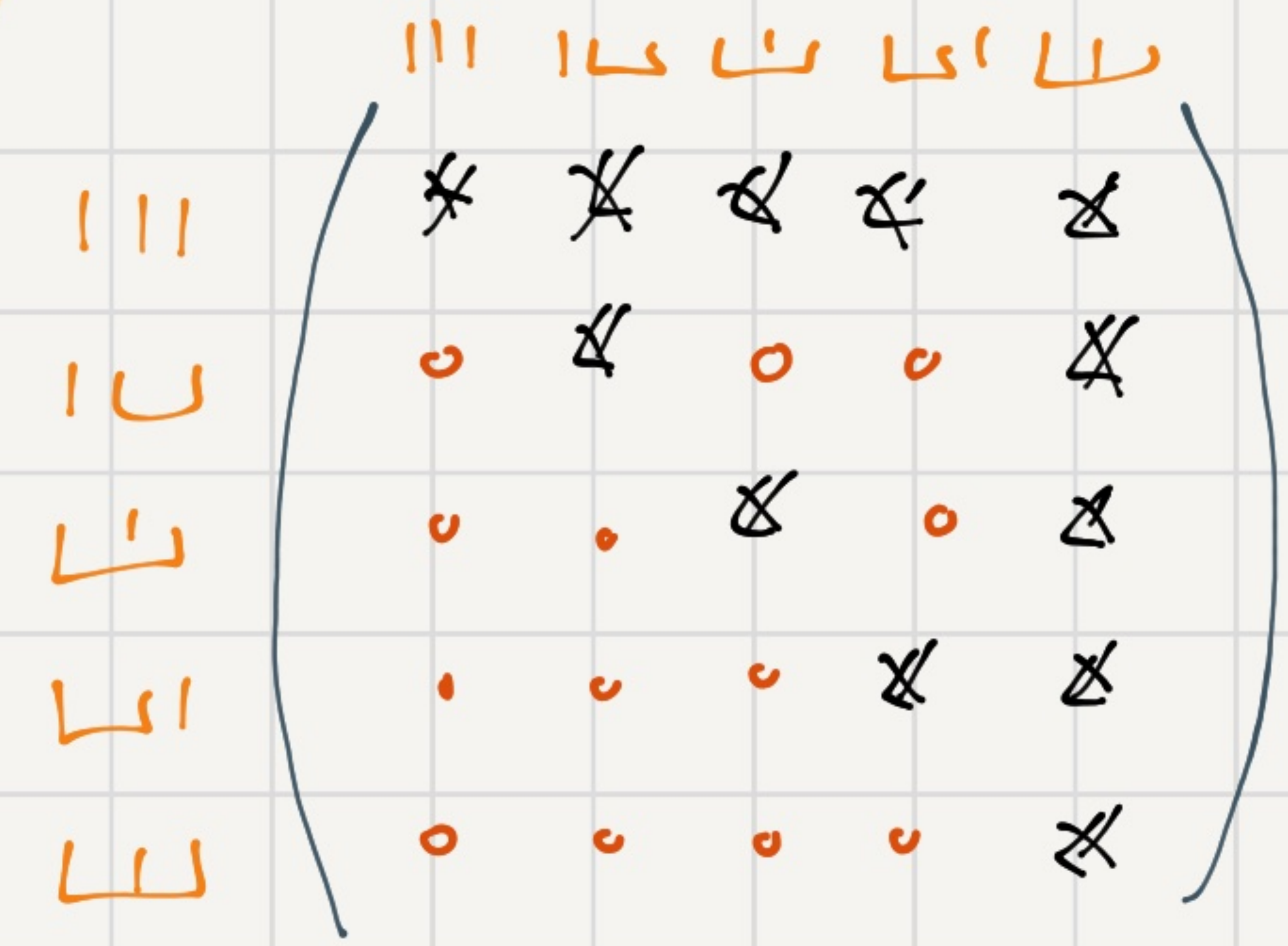
Note.

$$\mu * \zeta = \delta \iff \sum_{x \leq z \leq y} \mu(x,z) \overbrace{\cdot \zeta(z,y)} = \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}$$

$$\zeta * \mu = \delta \iff \sum_{x \leq z \leq y} \overbrace{\zeta(x,z) \cdot} \mu(z,y) = \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}$$

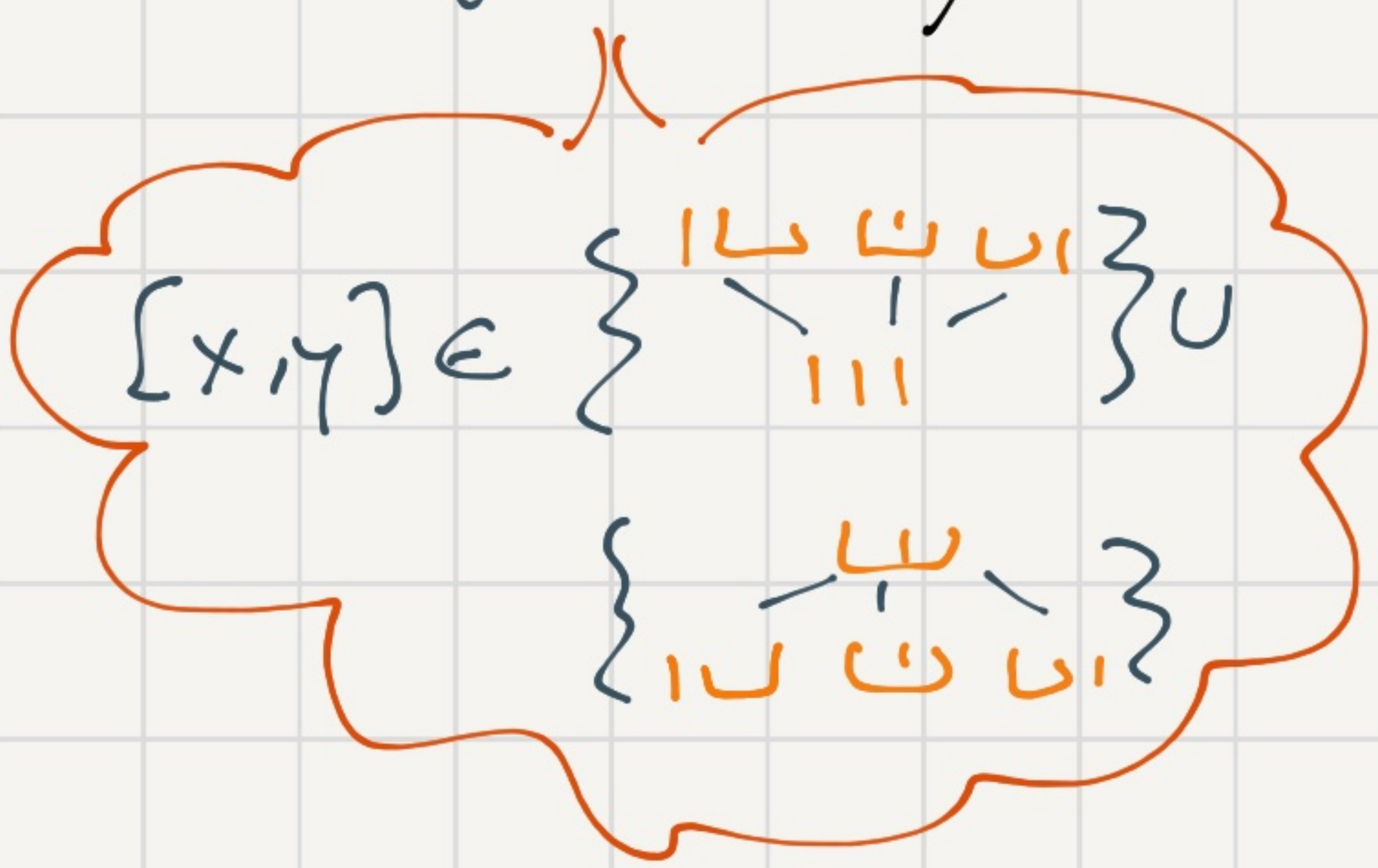
Example. $\mu_{NC(3)}$

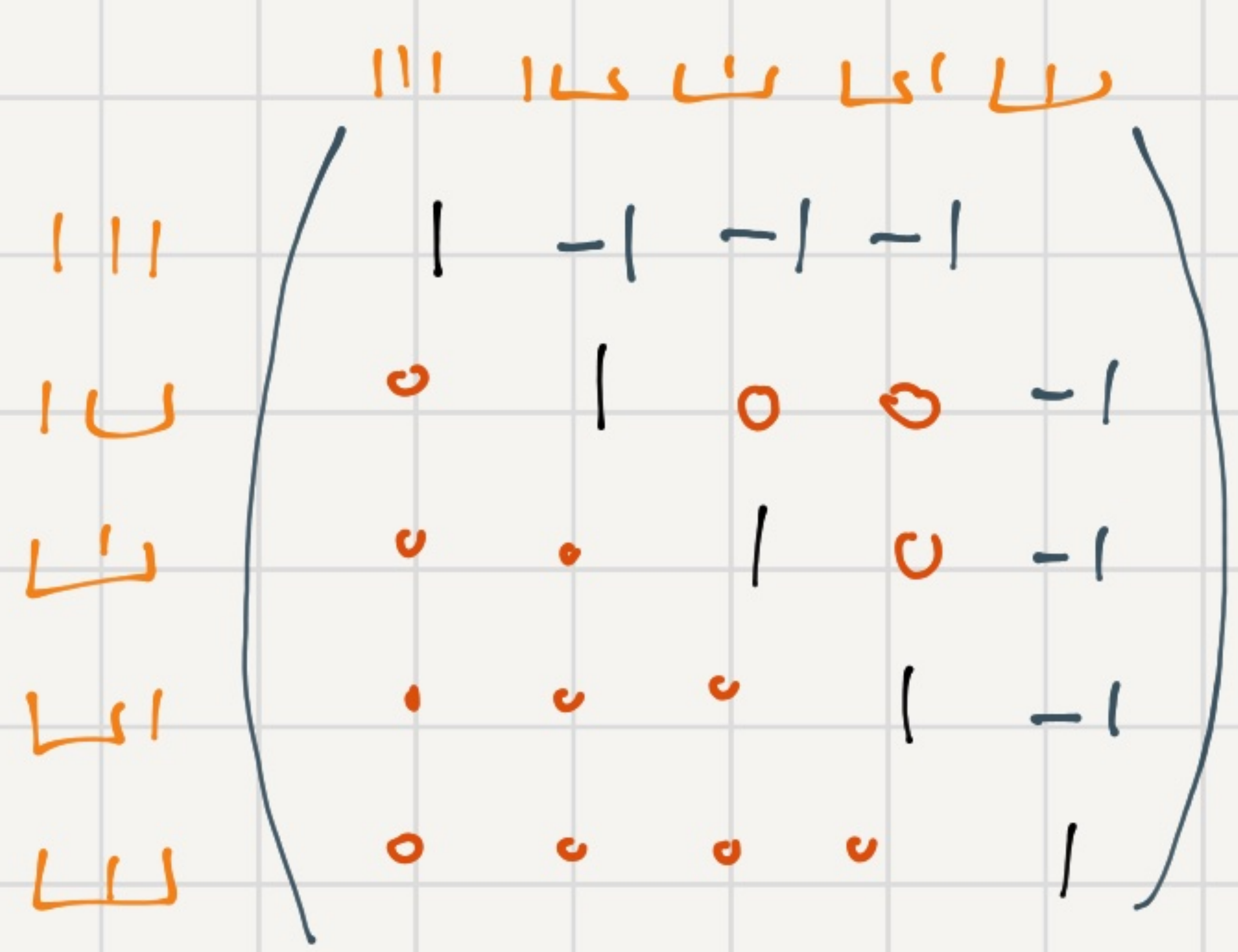
Length 0.

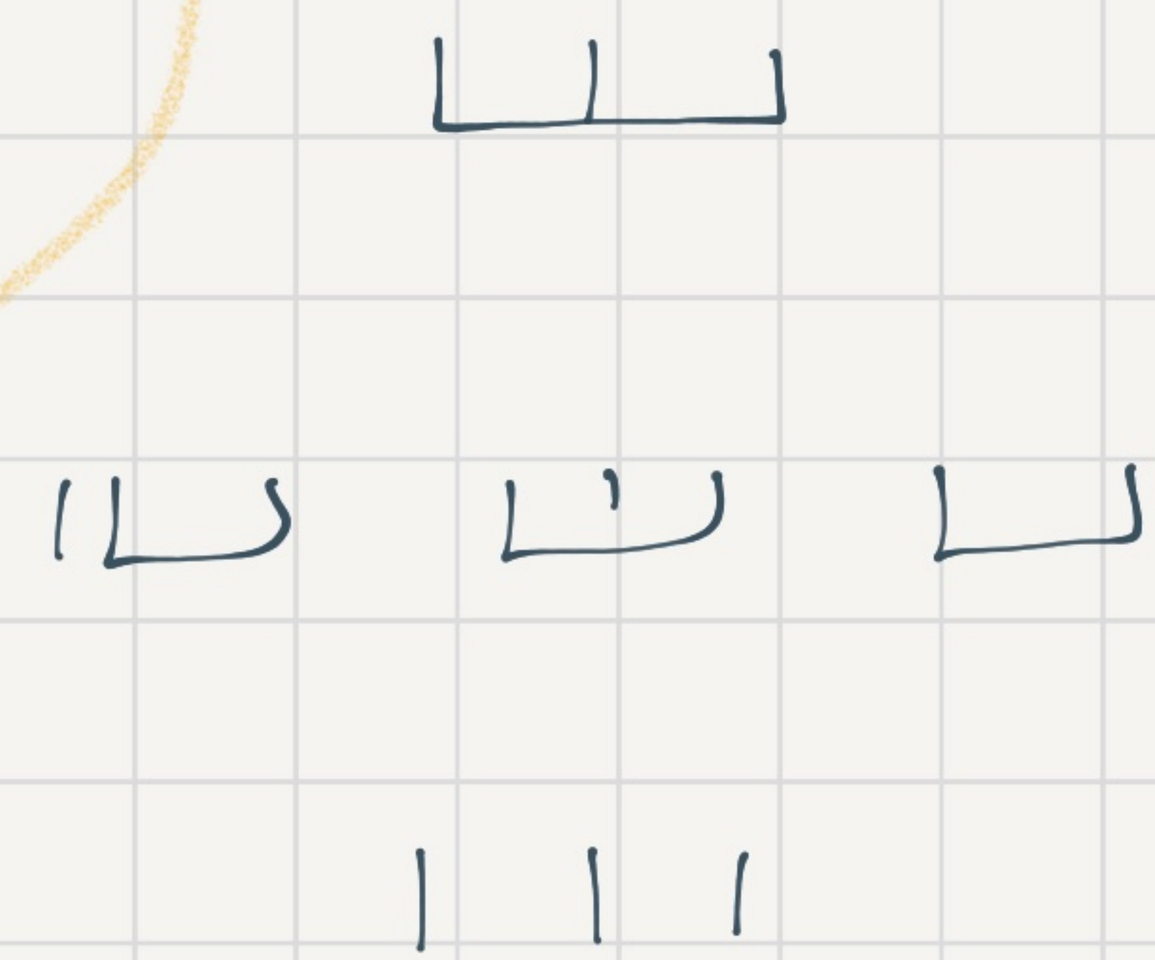


y "covers" x

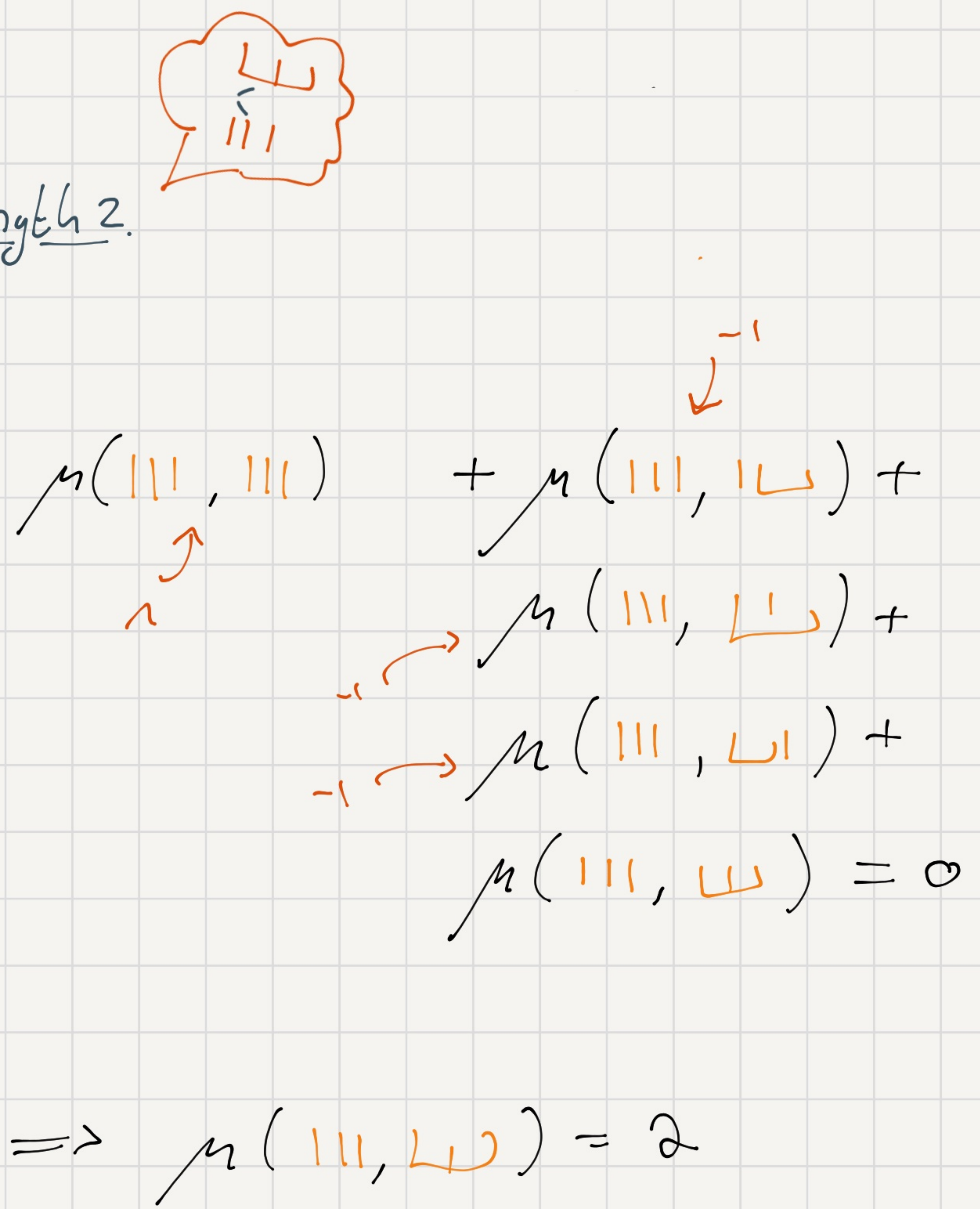
Length 1. $\mu(x,x) + \mu(x,y) = 0$

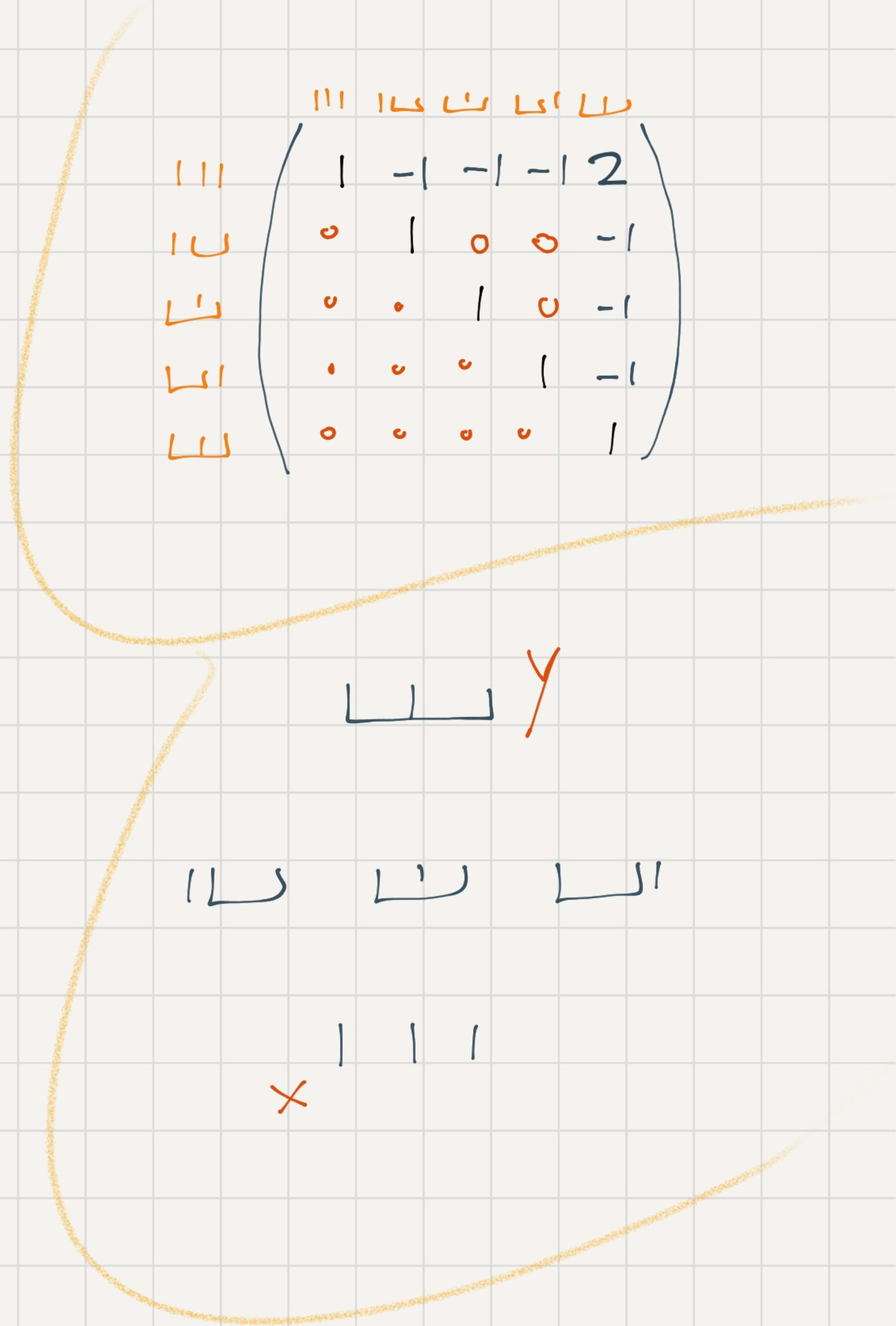






Length 2.





Back to one variable

Def. Let P be a finite poset. Given $f : P \to \mathbb{C}$ & $G \in A(P)$ we define $f * G : P \to \mathbb{C}$ by

$G : P \times P \to \mathbb{C}$
$G(x,y) \neq 0 \Rightarrow x \leq y$

$$f * G(x) = \sum_{y \leq x} f(y)\, G(y,x)$$

Note. From the view point of matrices, we can identify f with a (row) vector $\in \mathbb{C}^{|P|}$ & then $*$ corresponds to fG. Thus we have kind of associativity:

$$(f * G) * H = f * (G * H).$$

Note further that for $f, g : P \to \mathbb{C}$ the following are equivalent:

$$\forall x \in P \quad f(x) = \sum_{y \leq x} \underbrace{g(y)}_{\cdot \zeta(y,x)} \iff f = g * \zeta$$

$$\Updownarrow$$

$$\forall x \in P \quad g(x) = \sum_{y \leq x} f(y)\mu(y,x) \iff g = f * \mu$$

partial

Mobius inversion

in a lattice

For a lattice P, a "partial version" of the Mobius inversion formula holds:

Proposition. Let P be a finite lattice and $f, g : P \to \mathbb{C}$ s.t.

$$\forall \tau \qquad f(\tau) = \sum_{\pi \leq \tau} g(\pi)$$

Then,

check $\omega = 0/\tau$

$$\forall \omega \leq \tau \qquad \sum_{\omega \leq \sigma \leq \tau} f(\sigma)\mu(\sigma, \tau) = \sum_{\pi \vee \omega = \tau} g(\pi)$$

Pf.

$$\sum_{\omega \leq \sigma \leq \tau} f(\sigma)\mu(\sigma,\tau) = \sum_{\omega \leq \sigma \leq \tau} \sum_{\pi \leq \sigma} g(\pi)\mu(\sigma,\tau)$$

$$= \sum_{\pi \leq \tau} g(\pi) \sum_{\omega \vee \pi \leq \sigma \leq \tau} \mu(\sigma, \tau)$$

Fix $\pi \le \tau$. Recall that

$$\sum_{\omega \vee \pi \le \sigma \le \tau} \mu(\sigma, \tau) = \begin{cases} 1 & \omega \vee \pi = \tau \\ 0 & \omega \vee \pi < \tau \end{cases}$$

$$= \sum_{\omega \vee \pi \le \sigma \le \tau} \zeta(\omega \vee \pi, \sigma)\mu(\sigma, \tau) = \delta(\omega \vee \pi, \tau)$$

Thus,

$$\sum_{\omega \le \sigma \le \tau} f(\sigma)\mu(\sigma, \tau) = \sum_{\pi \vee \omega = \tau} g(\pi)$$

∎

Corollary. Let P be a finite lattice. Then, $\forall\, \omega \neq 0$

$$\sum_{\pi \vee \omega = 1} \mu(\sigma, \pi) = 0$$

who cares?

smaller ω $\Rightarrow$ fewer summands, one of which is $\mu(0,1)$. So this gives a "quick" way for computing $\mu(0,1)$

pf. Let $g : P \to \mathbb{C}$ defined by $g(\pi) = \mu(0, \pi)$.

Set $f = g * \zeta$. Then, $\forall\, \sigma \in P$

$$f(\sigma) = \sum_{\pi \leq \sigma} g(\pi) = \sum_{\pi \leq \sigma} \mu(0, \pi)\, \zeta(\pi, \sigma)$$

$$= (\mu * \zeta)(0, \sigma) = \begin{cases} 1 & \sigma = 0 \\ 0 & \sigma \neq 0 \end{cases}$$

f(σ) = 0 when σ ≠ 0

Previous Proposition

$$0 = \sum_{\omega \le \sigma \le \tau} f(\sigma)\mu(\sigma,\tau) = \sum_{\pi \vee \omega = \tau} g(\pi)$$

Setting $\tau = 1$ completes the proof. ∎

Example.

$$\sum_{\substack{\pi \vee 1|23 = 123 \\ \omega = 1|23}} \mu(1|2|3, \pi) = \overset{-1}{\mu(1|2|3,\ 12|3)} + \overset{-1}{\mu(1|2|3,\ 13|2)} + \overset{2}{\mu(1|2|3,\ 123)}$$

$$= 0$$

$$\begin{array}{c|c}
 & \begin{array}{ccccc} 1|2|3 & 1|23 & 13|2 & 12|3 & 123 \end{array} \\
\begin{array}{c} 1|2|3 \\ 1|23 \\ 13|2 \\ 12|3 \\ 123 \end{array} &
\begin{pmatrix}
1 & -1 & -1 & -1 & 2 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}$$

The Mobius function of NC

Proposition. If P, Q are finite posets then

$$\mu_{P\times Q}((p_1,q_1),(p_2,q_2)) = \mu_P(p_1,p_2)\cdot\mu_Q(q_1,q_2)$$

Bonus example before computing μ_{NC}..

Consider $(\mathbb{N}, \cdot|\cdot)$. Given $m|n$, $[m,n] \cong [1, \frac{n}{m}]$, so lets consider $\mu(1,n)$. If $n = p_1^{e_1}\cdots p_r^{e_r}$ then

$$[1,n] \cong [0,e_1]\times\cdots\times[0,e_r]$$

usual (chain) order

Recall $\mu_c(i,j) = \begin{cases} 1 & j=i \\ -1 & j=i+1 \\ 0 & \text{o.w.} \end{cases}$. Thus,

c for chain

$$\mu(1,n) = \mu_c(0,e_1)\cdots\mu_c(0,e_r) = \begin{cases} 0 & \text{if } \exists e_i > 1 \\ (-1)^r & \text{o.w.} \end{cases}$$

Denote the Mobius function of $NC(n)$ by μ_n and let

$$s_n = \mu_n(0,1)$$

We saw
$s_1 = 1$
$s_2 = -1$
$s_3 = 2$

Say we wish to compute $\mu(\pi,\sigma)$. Using our factorization result

$$[\pi,\sigma] = NC(1)^{k_1} \times \cdots \times NC(n)^{k_n}$$

Thus,

$$\mu(\pi,\sigma) = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$$

So it suffices to compute the s_n-s.

Proposition. $s_n = (-1)^{n-1} C_{n-1}$

pf. Indeed holds for $n = 1, 2, 3$ so assume $n \geq 4$. Take

$\omega =$ 

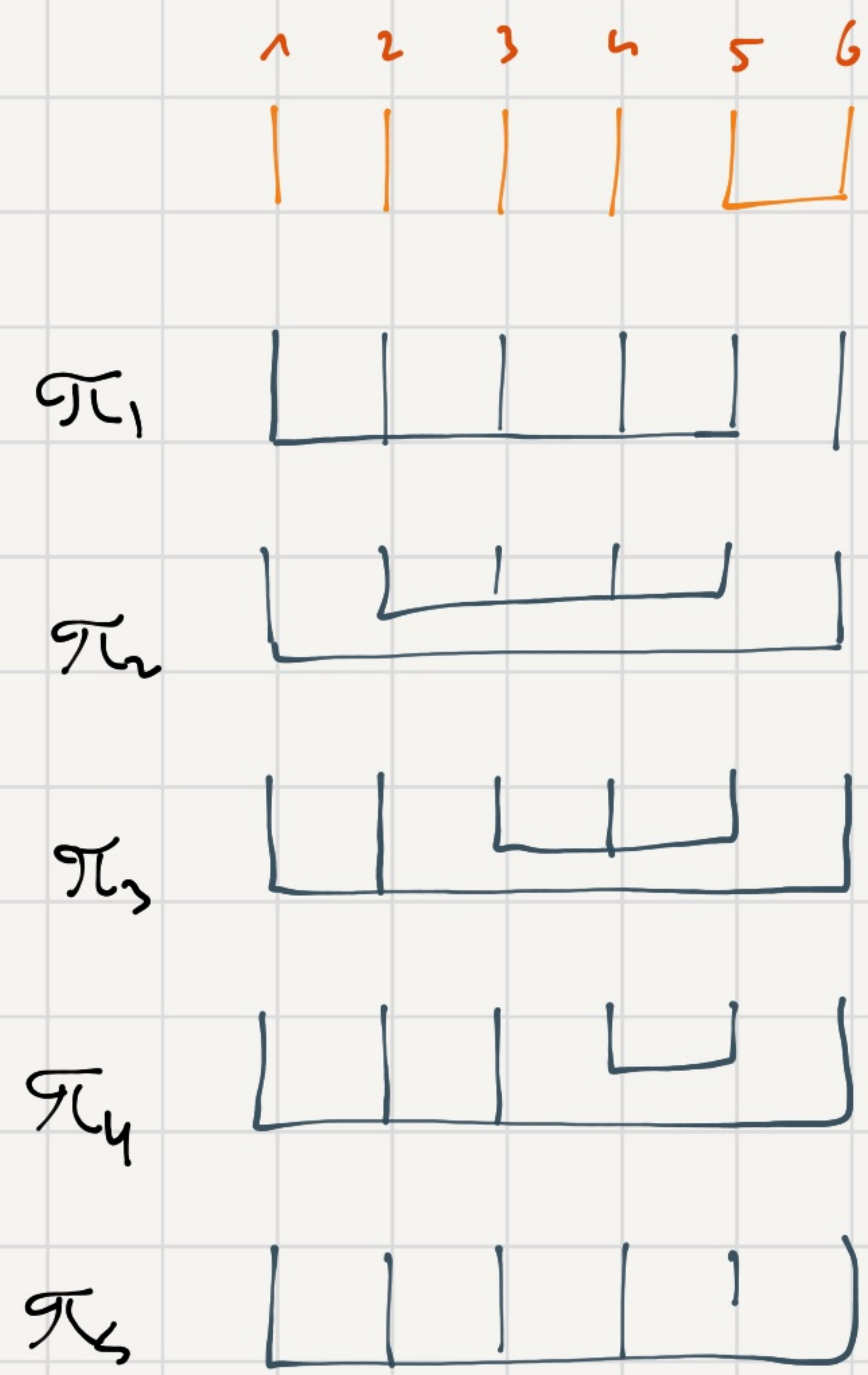
and apply the previous corollary

$$\sum_{\pi \vee \omega = \text{[diagram]}} \mu(\sigma, \pi) = 0$$

Now, the π-s that satisfy $\pi \vee \omega =$  are

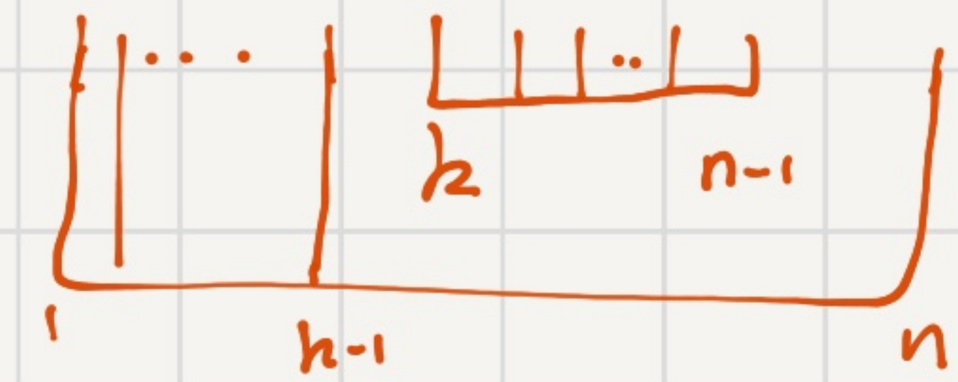
$\pi_k = \{\{k, \ldots, n-1\}, \{1, \ldots, k-1, n\}\}$
$k \geq 2$

(& even $k=1$ as long as we interpret $\{1, \ldots, k-1, n\}$ as $\{n\}$)



(&  of course)

So $$0 = \underbrace{\mu_n(0,1)}_{s_n} + \sum_{k=1}^{n-1} \mu_n(0,\pi_k)$$

$\pi_k =$ 

$$[0,\pi_k] \cong NC(k) \times NC(n-k)$$

Thus, $$0 = s_n + \sum_{k=1}^{n-1} s_k s_{n-k}$$

Set $t_n = (-1)^n s_{n+1}$ then $$0 = t_{n-1} - \sum_{k=1}^{n-1} t_{k-1} t_{n-k-1}$$

Thus, taking n instead of $n-1$:

$$t_n = \sum_{k=1}^{n} t_{k-1} t_{n-k}$$

which together with the matching initial values yield $t_n = c_n$. ∎