Random Walks on Rotating Expanders

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Joint work with Gal Maor (Tel Aviv University)

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Spectral Graph Theory 101

Spectral graph theory studies graphs by looking at the spectrum of related matrices.

G will be a d-regular undirected graph on n vertices with adjacency matrix $\mathbf{A}$ whose eigenvalues are

$$-d \leq \lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1 = d.$$

Spectral Graph Theory 101

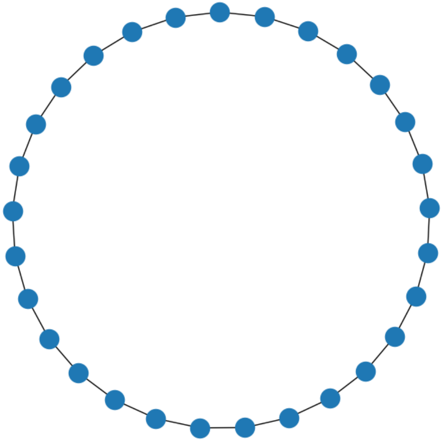

Figure: The 30-vertex cycle.

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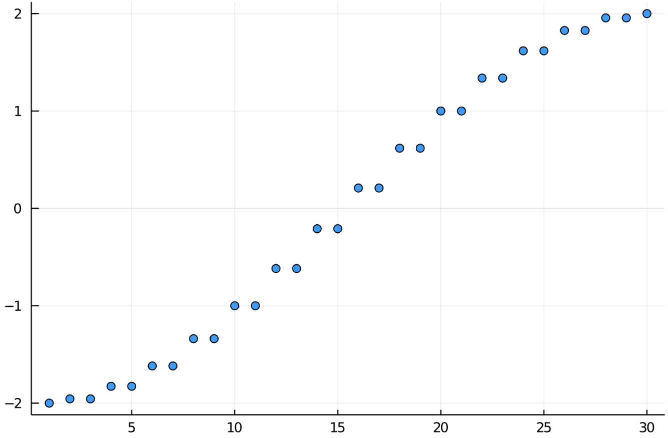

Figure: The spectrum of the 30-vertex cycle.

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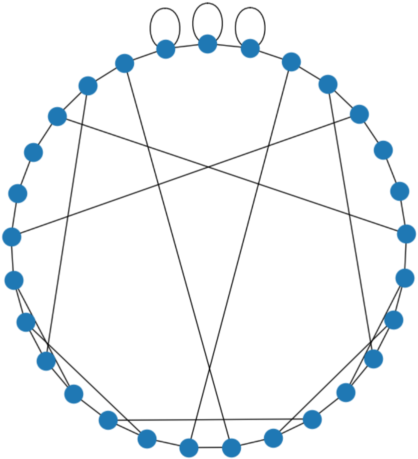

Figure: The 29-vertex cycle & inverses graph.

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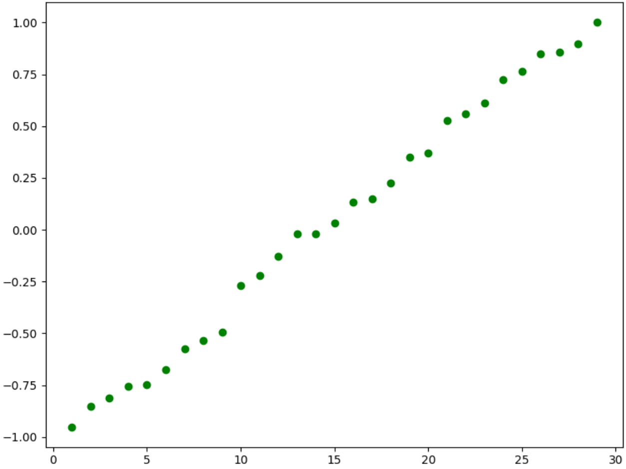

Spectral Graph Theory 101

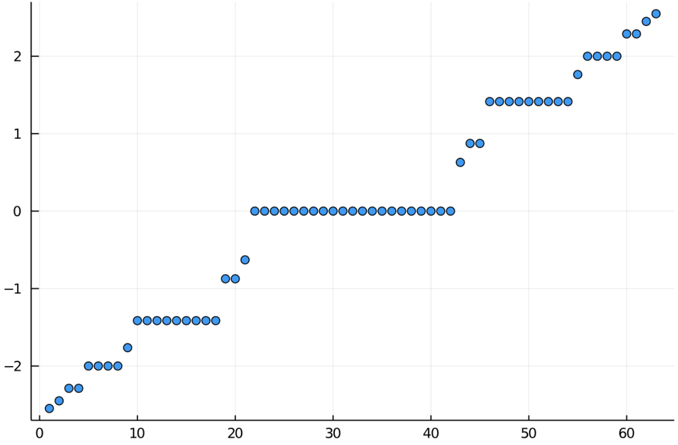

Figure: Who am I?

Spectral Graph Theory 101

$$-d \leq \lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1 = d.$$

The spectral expansion of G is given by

$$\lambda = \max\left(|\lambda_2|, |\lambda_n|\right).$$

The smaller λ is - the better. E.g., random walks mix quickly.

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Random walks are analyzed by studying $\mathbf{A}^t$ which corresponds to G^t that has degree $D = d^t$ and spectral expansion λ^t.

Spectral Graph Theory 101

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A lot of beautiful work has been done:

Lubotzky, Phillips, Sarnak'88 and Margulis'88, Bilu-Linial'06, Marcus, Spielman, Srivastava'15, Cohen'16, and many others.

Nearly-Ramanujan graphs have been studied by e.g., Friedman'08, Reingold,Vadhan and Wigderson'02, Ben-Aroya and Ta-Shma'15, Bordenave'19, Mohanty, O'Donnell and Paredes'19, and Alon'20.

A lot of **combinatorics**, **number theory**, **group theory**, and a little bit of **analysis**.

Overview

A downside to expander random walks

Take a d-regular Ramanujan graph G. Then,

$$\lambda = 2\sqrt{d-1} \approx 2\sqrt{d}.$$

So G^t has degree $D = d^t$ and spectral expansion

$$\lambda^t \approx (2\sqrt{d})^t = 2^t\sqrt{D} = 2^{t-1} \cdot 2\sqrt{D}.$$

So the spectral expansion deteriorates exponentially in t when compared to a Ramanujan graph of the same degree.

A downside to expander random walks

Question. Can we avoid this exponential loss?

Answer. Strictly speaking, no.

Still... Can we do something similar to a random walk that has slower deterioration in t?

Overview

Step-permute-step

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1. take the first step according to G;
2. permute G-s vertices, and take the second step according to the permuted G.

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We prefer to end up with an undirected graph, so we consider

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For technical reasons, we in fact study

$$\mathbf{A}_{\mathbf{P}} \triangleq \mathbf{A} \cdot \mathbf{PA}^2\mathbf{P}^{\mathsf{T}} \cdot \mathbf{A}.$$

Step-permute-step

$$\mathbf{A}_{\mathbf{P}} \triangleq \mathbf{A} \cdot \mathbf{P}\mathbf{A}^2\mathbf{P}^{\mathsf{T}} \cdot \mathbf{A}.$$

$\mathbf{A}_{\mathbf{P}}$ corresponds to a $D = d^4$ regular graph. Assuming G is Ramanujan,

$$2\sqrt{D} \leq \lambda(\mathbf{A}_{\mathbf{P}}) \leq 16\sqrt{D}.$$

Question. What does one permutation buy us?

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We prove that for every sufficiently high-girth G,

$$\exists \mathbf{P} = \mathbf{P}(G) \quad \text{s.t.} \quad \lambda(\mathbf{A}_{\mathbf{P}}) \leq \frac{27}{4}\sqrt{D} + o(1) < 7\sqrt{D}.$$

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Simulations suggest that $\frac{27}{4}$ is the typical behavior.

Our results

Theorem (Main result on Ramanujan graphs; informal)

By carefully permuting the vertices between steps one can achieve a linear deterioration in t.

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$\forall d$-regular Ramanujan graph G and $t \geq 2$ $\exists$ a sequence of permutation matrices $\mathbf{P} = (\mathbf{P}_1, \ldots, \mathbf{P}_{t-1})$ s.t.

$$\lambda(\mathbf{A}_\mathbf{P}) \leq \left(1 + \frac{1}{t}\right)^t (t+1)\sqrt{D} + \varepsilon < e(t+1)\sqrt{D} + \varepsilon,$$

where $D = d^{2t}$ and

$$\mathbf{A}_\mathbf{P} = \mathbf{A}\mathbf{P}_{t-1}\mathbf{A} \cdots \mathbf{P}_2\mathbf{A}\mathbf{P}_1\mathbf{A}^2\mathbf{P}_1^\top\mathbf{A}\mathbf{P}_2^\top \cdots \mathbf{A}\mathbf{P}_{t-1}^\top\mathbf{A},$$

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and $\varepsilon = 2^{-\Omega(g)}(td)^t$.

Our results

Theorem (Main result; informal)

- If t is not too large one pays λ for the first step only, and $1 \cdot \sqrt{d}$ for every other step;
- For large t, one get a linear deterioration with t, independent of λ.

Theorem (Main result)

$\forall d$-regular λ-spectral expander G and $t \geq 2$ $\exists$ a sequence of permutation matrices $\mathbf{P}$ s.t.

$$\lambda\left(\mathbf{A}_{\mathbf{P}}\right) \leq \begin{cases} O(\lambda^2 d^{t-1}) = O(\lambda\sqrt{D}), & t < \frac{\lambda^2}{d}; \\ e(t+1)\sqrt{D} + \varepsilon, & \text{otherwise.} \end{cases}$$

Overview

Rotating the expander?

Expand

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \psi_i \psi_i^{\mathsf{T}},$$

where $\psi_1, \ldots, \psi_n$ are respective eigenvectors. So

$$\mathbf{A}^t = \left(\sum_{i=1}^{n} \lambda_i \psi_i \psi_i^{\mathsf{T}} \right)^t = \sum_{i=1}^{n} \lambda_i^t \psi_i \psi_i^{\mathsf{T}}.$$

That is, the eigenvector ψ_2 is aligned, well, with itself.

Rotating the expander?

Can we break the alignments by rotation?

Rotating the expander?

Can we break the alignments by rotation?

Let's play lose with the graph structure, and rotate the eigenvectors. So instead of $\mathbf{A}^t$ we may consider

$$\prod_{i=1}^{t} \mathbf{Q}_i \mathbf{A} \mathbf{Q}_i^{\mathsf{T}},$$

where the $\mathbf{Q}_i$-s are orthogonal matrices.

Rotating the expander?

We rather end up with an undirected graph, so we consider

$$\mathbf{A}_{\mathbf{Q}} = \mathbf{A}\mathbf{Q}_{t-1}\mathbf{A}\cdots\mathbf{Q}_2\mathbf{A}\mathbf{Q}_1\mathbf{A}^2\mathbf{Q}_1^{\top}\mathbf{A}\mathbf{Q}_2^{\top}\cdots\mathbf{A}\mathbf{Q}_{t-1}^{\top}\mathbf{A}.$$

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A natural first step is to consider

$$\mathbb{E}_{\mathbf{Q}}\|\mathbf{A_Q}\|$$

where $\mathbf{Q}_1, \ldots, \mathbf{Q}_{t-1}$ are independent Haar random orthogonal matrices.

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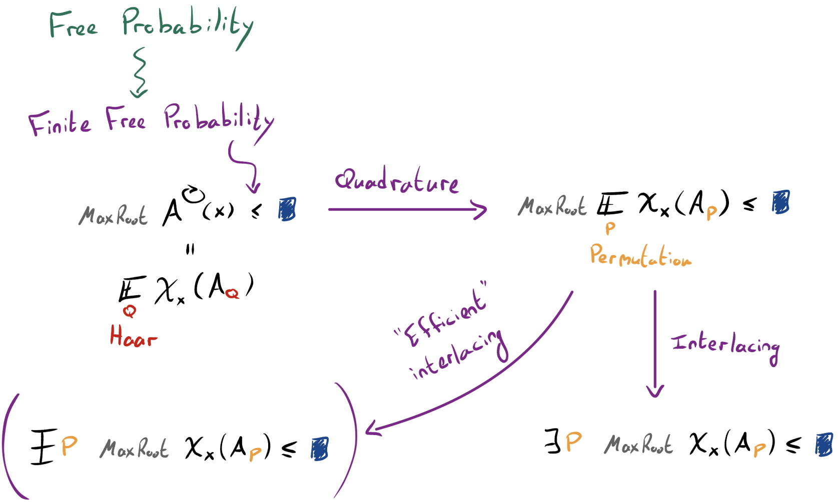
Inspired by the (fantastic!) works of Marcus, Spielman and Srivastava, we track the entire spectrum by attempting to bound the max-root of

$$\mathbf{A}^{\circlearrowleft}(x) = \mathbb{E}_{\mathbf{Q}}\left[\chi_x(\mathbf{A_Q})\right].$$

Road map



Free Probability
Finite Free Probability
Quadrature
MaxRoot $A^{\circlearrowright}(x) \leq$
=
$\mathbb{E}_{Q} \chi_x(A_Q)$
Haar
MaxRoot $\mathbb{E}_{P} \chi_x(A_P) \leq$
Permutation
"Efficient" interlacing
Interlacing
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Overview

The multiplicative convolution

Definition (Multiplicative convolution)

Let $\mathbf{A}, \mathbf{B}$ be real symmetric matrices with characteristic polynomials $a(x), b(x)$. The multiplicative convolution $a \boxtimes b$ is defined by

$$(a \boxtimes b)(x) = \mathbb{E}_{\mathbf{Q}}[\chi_x(\mathbf{AQBQ}^\mathsf{T})],$$

where $\mathbf{Q}$ is Haar random orthogonal matrix.

The multiplicative convolution

$$(a \boxtimes b)(x) = \mathbb{E}_{\mathbf{Q}}[\chi_x(\mathbf{AQBQ}^\top)]$$

In our case, for $t = 2$,

$$\mathbf{A}_{\mathbf{Q}} = \mathbf{AQA}^2\mathbf{Q}^\top\mathbf{A},$$

and so

$$\chi_x(\mathbf{A}_{\mathbf{Q}}) = \chi_x(\mathbf{AQA}^2\mathbf{Q}^\top\mathbf{A}) = \chi_x(\mathbf{A}^2\mathbf{QA}^2\mathbf{Q}^\top).$$

Thus,

$$\mathbb{E}_{\mathbf{Q}}\chi_x(\mathbf{A}_{\mathbf{Q}}) = \chi_x(\mathbf{A}^2) \boxtimes \chi_x(\mathbf{A}^2).$$

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Thus,

$$\mathbb{E}_{\mathbf{Q}}\chi_x(\mathbf{A}_{\mathbf{Q}}) = \chi_x(\mathbf{A}^2) \boxtimes \chi_x(\mathbf{A}^2).$$

Similarly, for every $t \geq 2$,

$$\mathbb{E}_{\mathbf{Q}}\chi_x(\mathbf{A}_{\mathbf{Q}}) = \chi_x(\mathbf{A}^2)^{\boxtimes t}.$$

The $\mathcal{M}$ and $\mathcal{N}$ transforms

Let μ be a distribution on $[0, a]$. Define

$$\mathcal{M}_\mu(x) = \int_0^a \frac{t}{x-t}\mu(t)dt$$

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We extend $\mathcal{M}_\mu$ to real-rooted polynomials $p(x)$ and then to real symmetric matrices in the natural way. E.g.,

$$\mathcal{M}_{\mathbf{A}}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{\lambda_i}{x-\lambda_i}.$$

Lastly, $\mathcal{N}_\mu(y)$ is the largest x s.t. $\mathcal{M}_\mu(x) = y$.

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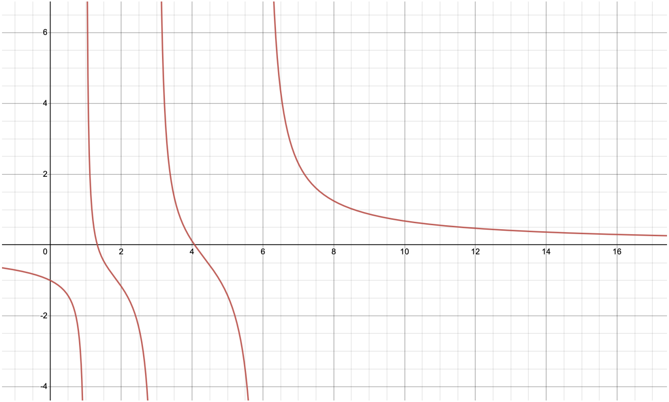

Observe that for every $y > 0$, $\mathcal{N}_{\mathbf{A}}(y)$ is an upper bound on λ_1.

The multiplicative convolution and the $\mathcal{N}$ transform

We wish to bound the largest root of

$$\mathbb{E}_{\mathbf{Q}} \chi_x(\mathbf{A}_{\mathbf{Q}}) = \chi_x(\mathbf{A}^2)^{\boxtimes t}.$$

Thus, we want to have a good bound on $\mathcal{N}_{\chi_x(\mathbf{A}^2)^{\boxtimes t}}$.

The multiplicative convolution and the $\mathcal{N}$ transform

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Theorem (MSS'15)

$\forall p(x), q(x)$ with non-negative real roots and every $y > 0$,

$$\mathcal{N}_{p\boxtimes q}(y) \leq \frac{y}{y+1} \cdot \mathcal{N}_p(y) \cdot \mathcal{N}_q(y).$$

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Hence,

$$\mathcal{N}_{\chi_x(\mathbf{A}^2)^{\boxtimes t}}(y) \leq \left(\frac{y}{y+1}\right)^{t-1} \mathcal{N}_{\mathbf{A}^2}(y)^t.$$

Back to our analysis

$$\mathcal{N}_{\chi_x(\mathbf{A}^2)^{\boxtimes t}}(y) \le \left(\frac{y}{y+1}\right)^{t-1} \mathcal{N}_{\mathbf{A}^2}(y)^t.$$

$\mathcal{N}_{\mathbf{A}^2}(y)$ is difficult to work out as it is the max-inverse of

$$\mathcal{M}_{\mathbf{A}^2}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{\lambda_i^2}{x-\lambda_i^2}.$$

Cheat!

Replace Spec $\mathbf{A}$ with the Kesten-McKay distribution.

$$\mathcal{N}_{\mathbf{A}^2}(y) \rightsquigarrow \mathcal{N}_{\mathrm{km}^2}(y)$$

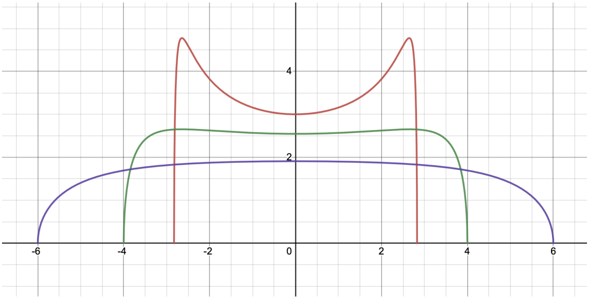

Figure: The Kesten-McKay distribution for $d = 3, 5, 10$.

Let's calculate

Recall that $\mathcal{N}_{\mathbf{A}^2}(y)$ is the max-inverse of

$$\mathcal{M}_{\mathbf{A}^2}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{\lambda_i^2}{x - \lambda_i^2},$$

Instead, we will work with the km^2 distribution.

$$\mathcal{M}_{\mathrm{km}^2}(x) = \frac{2d}{x - 2d + \sqrt{x^2 - 4(d-1)x}},$$

$$\mathcal{N}_{\mathrm{km}^2}(y) = \frac{d^2(y+1)^2}{y(y+d)}.$$

Let's calculate

$$\mathcal{N}_{\mathrm{km}^2}(y) = \frac{d^2(y+1)^2}{y(y+d)}.$$

Hence

$$\begin{aligned}\mathcal{N}_{(\mathrm{km}^2)^{\boxtimes t}}(y) &\le \left(\frac{y}{y+1}\right)^{t-1} \cdot \mathcal{N}_{\mathrm{km}^2}(y)^t \\ &= \left(\frac{d^2}{d+y}\right)^t \cdot \frac{(y+1)^{t+1}}{y}.\end{aligned}$$

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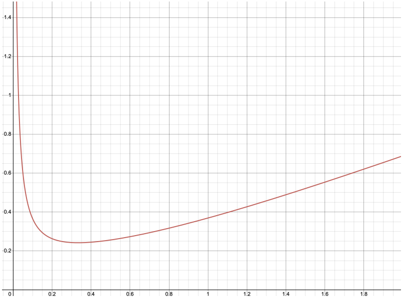
Ignoring the $\frac{1}{y}$ factor, we would have plugged $y = 0$ and get

$$d^t = \sqrt{D}.$$

Let's calculate

$$\mathcal{N}_{(\mathrm{km}^2)^{\boxtimes t}}(y) \leq \left(\frac{d^2}{d+y}\right)^t \cdot \frac{(y+1)^{t+1}}{y}.$$



$y_{\min} = \frac{d}{dt-t-1}$ minimizes the RHS, yielding

$$\mathcal{N}_{(\mathrm{km}^2)^{\boxtimes t}}(y_{\min}) = \left(1+\frac{1}{t}\right)^t (t+1)d^t < e(t+1)d^t.$$

Uncheat

Recall that

$$\mathcal{M}_{\mu}(x) = \sum_{r=1}^{\infty} \frac{m_r(\mu)}{x^r}.$$

To obtain the result about G we use the observation that the first $\frac{g}{2}$ moments of $\mathbf{A}$ and km are equal. Hence, for a large girth,

$$\mathcal{M}_{\mathbf{A}^2}(x) \approx \mathcal{M}_{\mathrm{km}^2}(x) \qquad \forall x \text{ sufficiently large.}$$

From this we can show that

$$\mathcal{N}_{\mathbf{A}^2}(y) \approx \mathcal{N}_{\mathrm{km}^2}(y) \qquad \forall y \text{ in some range.}$$

Summary

Summary. Rotate your expander while taking long random walks.

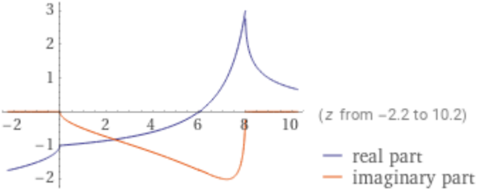
Many interesting questions!

1. Explicitness? Strongly explicitness?
2. Is the linear loss in t inherent?
3. Applications?
4. Other applications of finite free probability, quadrature and interlacing?

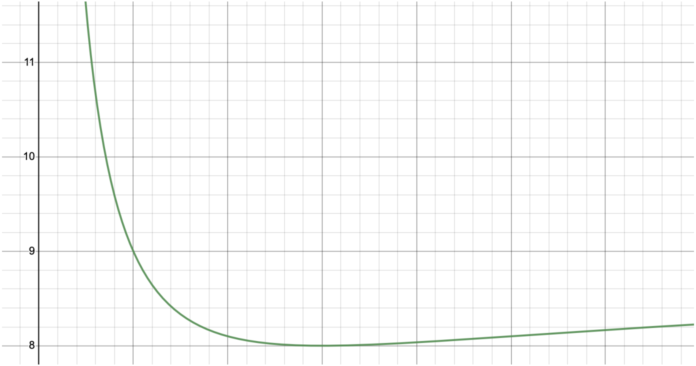
Thank you!

Some figures

$$\mathcal{M}_{\mathrm{km}^2}(x) = \frac{6}{x - 6 + \sqrt{x^2 - 8x}}$$



Some figures



$$\mathcal{N}_{\mathrm{km}^2}(y) = \frac{9(y+1)^2}{y(y+3)}$$