Measure Theory

101

Discussion. Recall that we wanted a distribution μ s.t.

$$\int z^k \bar{z}^\ell \, d\mu(z) = \frac{1}{d} \sum_{i=1}^{d} \lambda_i^k \bar{\lambda}_i^\ell$$

for some $\lambda_1, \dots, \lambda_d$.

Consider the special case in which, for some $p \in \mathbb{R}$

$$\int z^k \, d\mu(z) = p^k$$

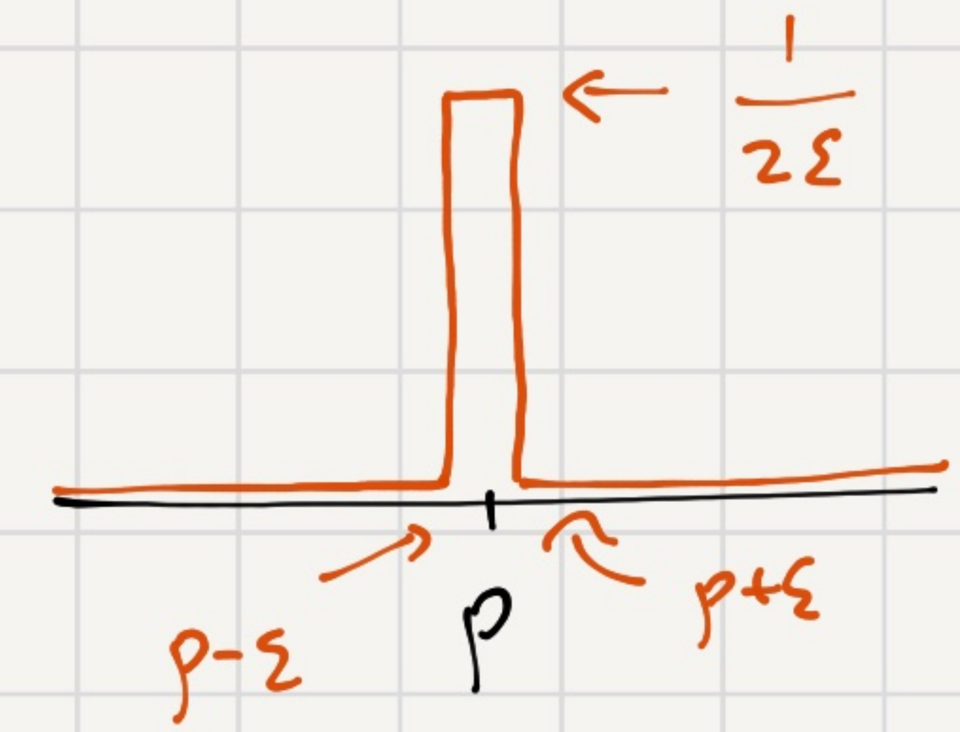
If we interpret the symbol "$d\mu(z)$" as $\mu(z)\,dz$
for some function $\mu(z)$ then we want

function

part of the integral...

$$\int z^h \mu(z) \, dz = p^k$$

Who is $\mu(z)$? We can take $\mu_\varepsilon(z) =$



Then $\displaystyle\int z^k \mu_\varepsilon(z)\,dz = \int_{p-\varepsilon}^{p+\varepsilon} z^k \cdot \frac{1}{2\varepsilon}\,dz$

$$= \frac{1}{2\varepsilon} \cdot \frac{1}{k+1} z^{k+1} \Big|_{z=p-\varepsilon}^{p+\varepsilon}$$

$$= \frac{1}{2\varepsilon} \frac{1}{k+1} \left((p+\varepsilon)^{k+1} - (p-\varepsilon)^{k+1} \right)$$

$$= \frac{1}{2\varepsilon} \frac{1}{k+1} \left(p^{k+1} + (k+1)p^k \varepsilon - p^{k-1} + (k+1)p^k \varepsilon + O(\varepsilon^2) \right)$$

$$= \frac{1}{2\varepsilon} \frac{1}{k+1} \cdot \left(2(k+1)p^k \varepsilon + O(\varepsilon^2) \right)$$

$$= p^k + O(\varepsilon)$$

So we really want a function which is ∞ at p & 0 o.w...

This is called the Dirac delta function but isn't really a function.

Solution The idea is not to wrongfuly think of $d\mu(z)$ as a shorthand for $\mu(z)dz$ for some function $\mu(z)$ but rather think of μ as a "measure" - a mathematical object which is not a function, we can integrate against.

$$\int \square \, d\mu$$

Integrating w.r.t the measure μ

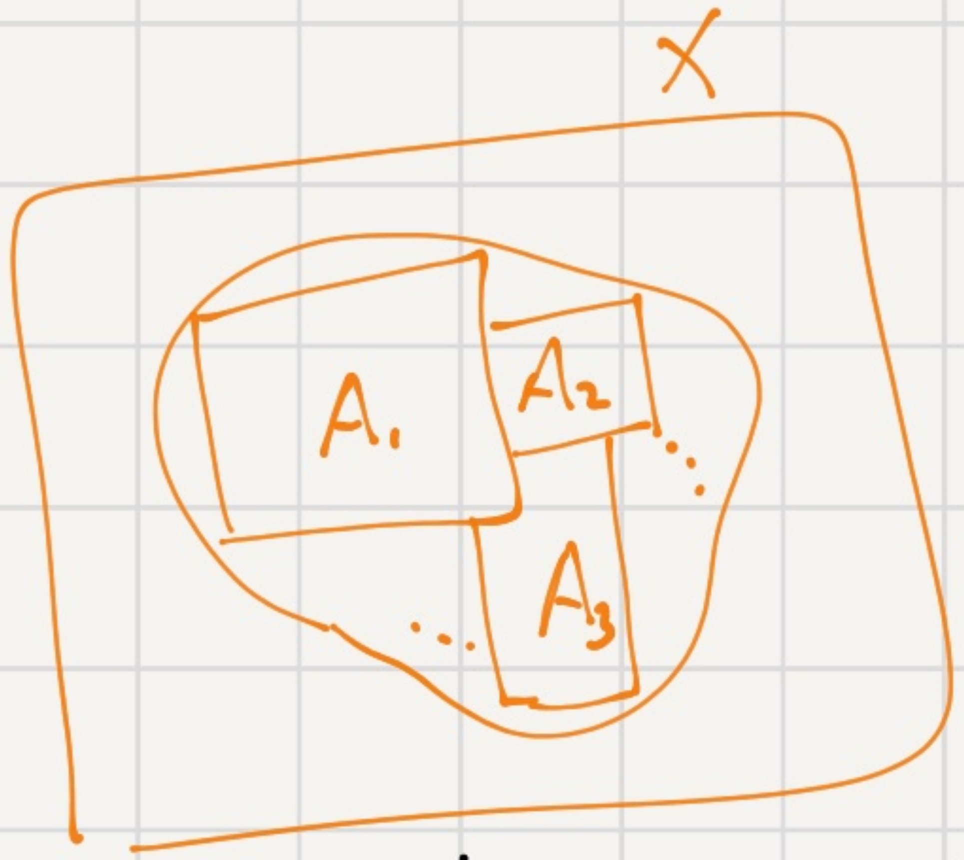
measure

is a linear functional ($\int f \, d\mu + \int g \, d\mu = \int (f+g) \, d\mu$, etc..)

Measurable Spaces

Def. Let $X^{(\neq \emptyset)}$ be a set. A set $\mathcal{A} \subseteq P(X)$ is a <u>σ-algebra</u> if:

* $X \in \mathcal{A}$
* $A \in \mathcal{A} \implies A^c \in \mathcal{A}$ — closed under complement — hence, $\emptyset \in \mathcal{A}$
* $A_i \in \mathcal{A} \quad i \in \mathbb{N} \implies \bigcup_i A_i \in \mathcal{A}$ — closed under countable unions



An element $A \in \mathcal{A}$ is called <u>$\mathcal{A}$-measurable</u> / <u>measurable set</u>

$(X, \mathcal{A})$ is called a <u>measurable space</u>.

Example.

* $\mathcal{A} = \{\emptyset, X\}$ — clearly, not interesting
* $\mathcal{A} = P(X)$ — not so useful when X is uncountable

<u>The point</u>

we typically can't assign a measure / volume of all subsets of an ambient set (such as $\mathbb{R}$).

Def. Let $\mathcal{M} \subseteq \mathcal{P}(X)$. Then, there is a smallest σ-algebra that contains $\mathcal{M}$:

w.r.t inclusion

Sometimes called the generating set

$$\sigma(\mathcal{M}) \triangleq \bigcap_{\substack{\mathcal{A} \supseteq \mathcal{M} \\ \mathcal{A}\ \sigma\text{-alg}}} \mathcal{A}$$

$\bigcap$ σ-alg is a σ-alg

$\mathcal{P}(X)$ participate in $\bigcap$

Example. Let $X = \mathbb{R}$. The Borel σ-algebra,

Can be defined for every topological space

$$\mathcal{B}(\mathbb{R}) = \sigma\left(\{(a,b) \mid a < b\}\right).$$

Includes thing like $[a,b)$, $[a,b]$, $\{a\}$

Measure

Spaces

Def. Let $(X, \mathcal{A})$ be a measurable space. A map

$$\mu : \mathcal{A} \longrightarrow [0, \infty]$$

$[0,\infty) \cup \{\infty\}$
∞ is to be considered as a symbol

is called a measure if

* $\mu(\emptyset) = 0$
* For $(A_i)_{i=1}^{\infty} \in \mathcal{A}$, $A_i \cap A_j = \emptyset$, $\quad \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

countable

σ-additivity

$(X, \mathcal{A}, \mu)$ is called a measure space.

A probability measure is a measure with $\mu(X) = 1$

Calculation with ∞. $\quad x+\infty \triangleq \infty \quad \forall x\in[0,\infty]$

$$x\cdot\infty \triangleq \infty \quad \forall x\in(0,\infty)$$

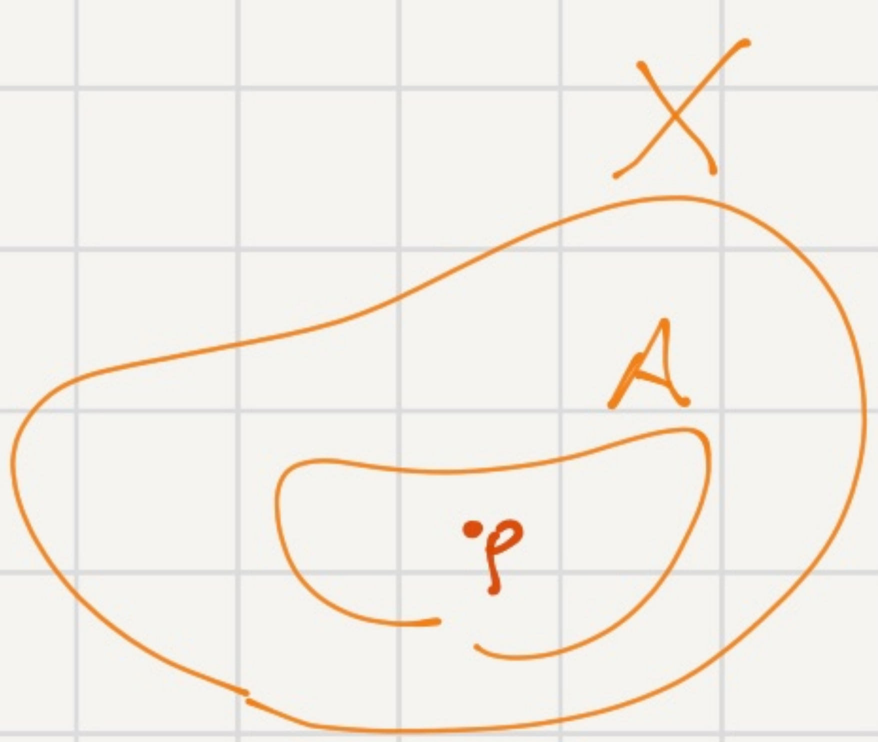
$$0\cdot\infty \begin{cases} \to \text{undefined} \leftarrow \text{us} \\ \to 0 \leftarrow \text{some people; not us} \end{cases}$$

Example. X, $\mathcal{A}=P(X)$. The counting measure is

$$\mu(A)=\begin{cases} |A| & A \text{ finite} \\ \infty & \text{o.w.} \end{cases}$$

Example. Let X be a set & $p\in X$. We define the Dirac measure for $p\in X$ on $\mathcal{A}=P(X)$ by

$$\delta_p(A)=\begin{cases} 1 & p\in A \\ 0 & p\notin A \end{cases}$$



$ We would like to find a measure on $X = \mathbb{R}$ s.t.

* $\mu([0,1]) = 1$
* $\mu(x + A) = \mu(A) \quad \forall x \in \mathbb{R}$

Turns out we cannot define such a measure on $P(\mathbb{R})$. But it can be defined on $\mathcal{B}(\mathbb{R})$:

can be defined for $\mathbb{R}^d$

Defthm. The Lebesgue measure on $\mathbb{R}$ is the unique measure

$$\lambda : \mathcal{B}(\mathbb{R}) \longrightarrow [0, \infty]$$

satisfying $\lambda((a,b)) = b - a. \qquad \forall a < b$

only measure on $\mathcal{B}(\mathbb{R})$ satisfying $

Measurable Maps

General Philosophy

Study spaces by studying maps between the spaces that preserve the structure

Def. Let $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$ be measurable spaces.

$f: \Omega_1 \to \Omega_2$ is <u>measurable</u> if

$$A_2 \in \mathcal{A}_2 \implies f^{-1}(A_2) \in \mathcal{A}_1.$$

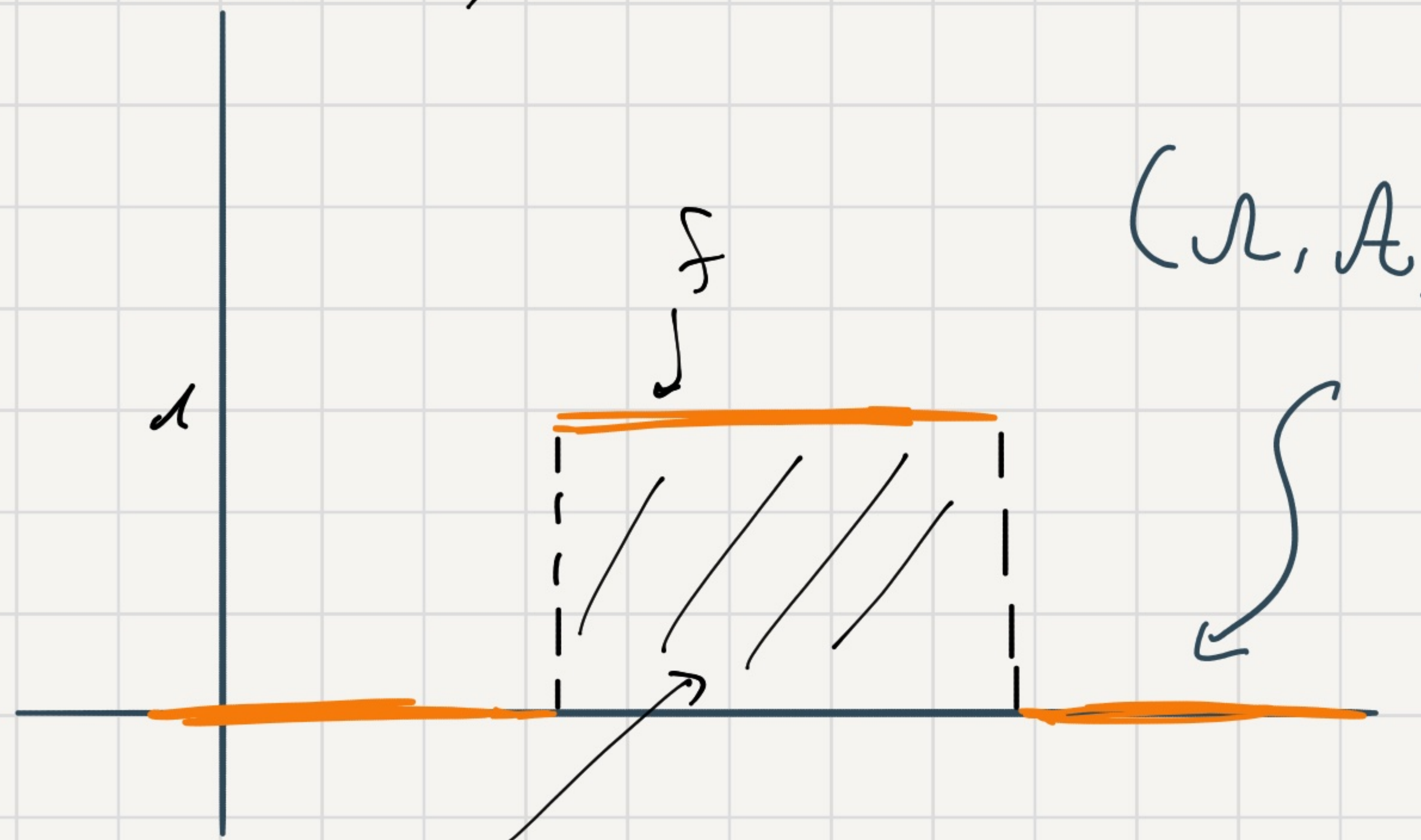
preimage of a measurable set is measurable

Borel σ-algebra

Lebesgue measure

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

$(\Omega, \mathcal{A}, \mu)$



$\mu(\underbrace{f^{-1}(\{1\})}_{})$ $\in \mathcal{A}$

Example. Let $(\Omega, \mathcal{A})$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The characteristic function

$$\chi_A : \Omega \to \mathbb{R} \qquad\qquad A \in \mathcal{A}$$
$$\omega \mapsto \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Then, χ_A is a measurable map:

$$\chi_A^{-1}(\underbrace{S}_{\in \mathcal{B}(\mathbb{R})}) = \begin{cases} \Omega & 0 \in S \wedge 1 \in S \\ \emptyset & 0 \notin S \wedge 1 \notin S \\ \Omega \setminus A & 0 \in S \wedge 1 \notin S \\ A & 0 \notin S \wedge 1 \in S \end{cases}$$

all four in $\mathcal{A}$

measurable space

Remark. Let $(\Omega, \mathcal{A})$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

If $f, g : \Omega \to \mathbb{R}$ are measurable then so are

$$\alpha f + \beta g \quad \forall \alpha, \beta \in \mathbb{R}$$

$$|f| \;, \quad f \cdot g \,, \; \dots \;, \quad \text{(also compositions)}.$$

Def. Let $(X, \mathcal{A})$ be a measurable space. A measurable function $f : X \to \mathbb{R}$ is called a measurable map.

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

E.g. $\chi_A : X \to \mathbb{R}$ $\quad A \in \mathcal{A}$.

Sets of measure 0

&

"almost everywhere"

Def. Let $(X, \mathcal{A}, \mu)$ be a measure space. $A \in \mathcal{A}$ is called a <u>set of measure zero</u> if $\mu(A) = 0$.

not only $\emptyset$ in general

Terminology. A condition $(=, \to, \dots)$ holds <u>almost always</u> / <u>almost everywhere</u> (a.e) if it holds except in a set N of measure zero.

Note that $\{x \mid f(x) \neq g(x)\}$ may not be measurable

E.g. $f, g : X \to Y$. We say $f = g$ a.e $f =_{a.e} g$ if

$(X, \mathcal{A}, \mu)$ any set

$\exists N \in \mathcal{A}$ of measure zero s.t. $\forall x \notin N$ $f(x) = g(x)$.

Lebesgue Integral

for non-negative measurable functions

will extend soon

Def. Let $(X, \mathcal{A}, \mu)$ be a measure space. for $A \in \mathcal{A}$ we define the Lebesgue integral of χ_A by

$$\int_X \chi_A \, d\mu \triangleq \mu(A)$$

∞ is perfectly valid

Def. Let $(X, \mathcal{A}, \mu)$ be a measure space. A measurable map

$f: X \to \mathbb{R}$, $\quad f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$ is called simple.

$c_i \in \mathbb{R}$

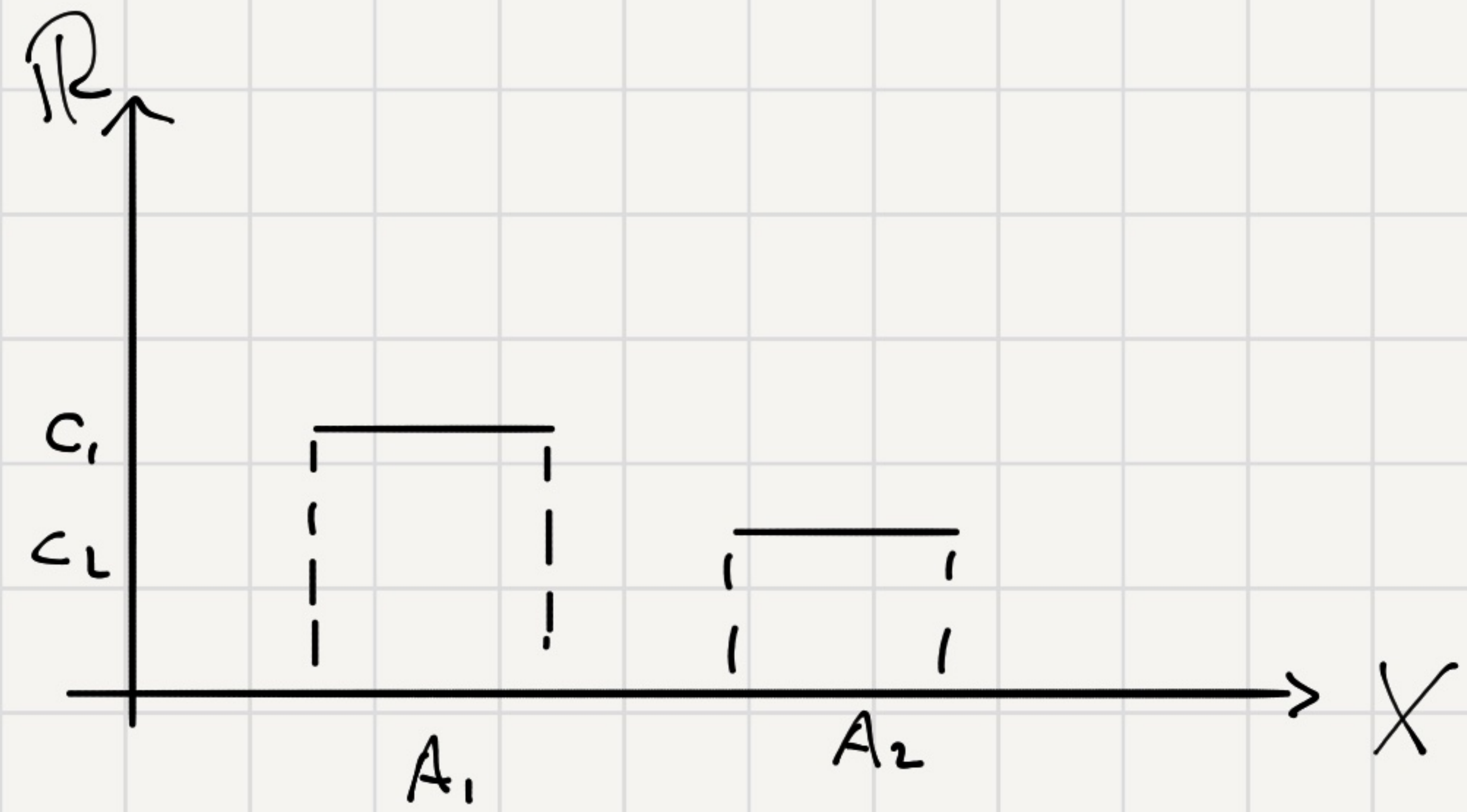
$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

measurable!

$\Updownarrow$

$|\operatorname{Im} f| < \infty$

Def. $S^+ = \{ f : X \to \mathbb{R} \text{ simple } \& \ f \geq 0 \}$



Def. For $f \in S^+$ the Lebesgue integral of $f = \sum_{i=1}^{n} c_i \chi_{A_i}$

wrt μ is given by

$$\int_X f \, d\mu \overset{!}{=} \sum_{i=1}^{n} c_i \mu(A_i) \in [0, \infty]$$

well-defined

Properties. $\forall f, g \in S^+$

* $\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu \quad \forall \alpha, \beta \geq 0$

* $f \overset{(=)}{\underset{ae}{\leq}} g \implies \int_X f\, d\mu \overset{(=)}{\leq} \int_X g\, d\mu$ (monotonicity)

* $\int f\, d\mu = 0 \iff f \underset{ae}{=} 0$

We would like to consider

$$f : X \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$$

$(X, \mathcal{A})$ $\quad$ $(\overline{\mathbb{R}}, ?)$

and talk about their measurability.

We define

We don't care about $\pm\infty$ for measurability

$$\mathcal{B}(\overline{\mathbb{R}}) = \{ A \subseteq \overline{\mathbb{R}} \mid A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R}) \}$$

Similarly, we'll restrict to $\overline{\mathbb{R}}_+ = [0, \infty) \cup \{\infty\}$ "=" $[0, \infty]$

and define $\mathcal{B}(\overline{\mathbb{R}}_+) = \{ A \subseteq \overline{\mathbb{R}}_+ \mid A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R}) \}$

Def. Let $f : X \to \overline{\mathbb{R}}_+$ measurable. We define

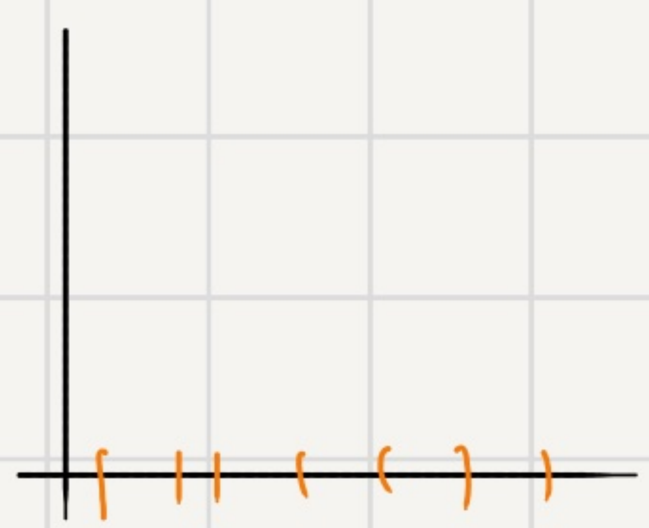
$(X, \mathcal{A}, \mu)$ $\quad (\overline{\mathbb{R}}_+, \mathcal{B}(\overline{\mathbb{R}}_+))$

$$\int_X f \, d\mu \triangleq \sup \left\{ \int_X h \, d\mu \;\middle|\; h \in S^+,\ h \leq f \right\} \in \overline{\mathbb{R}}_+$$

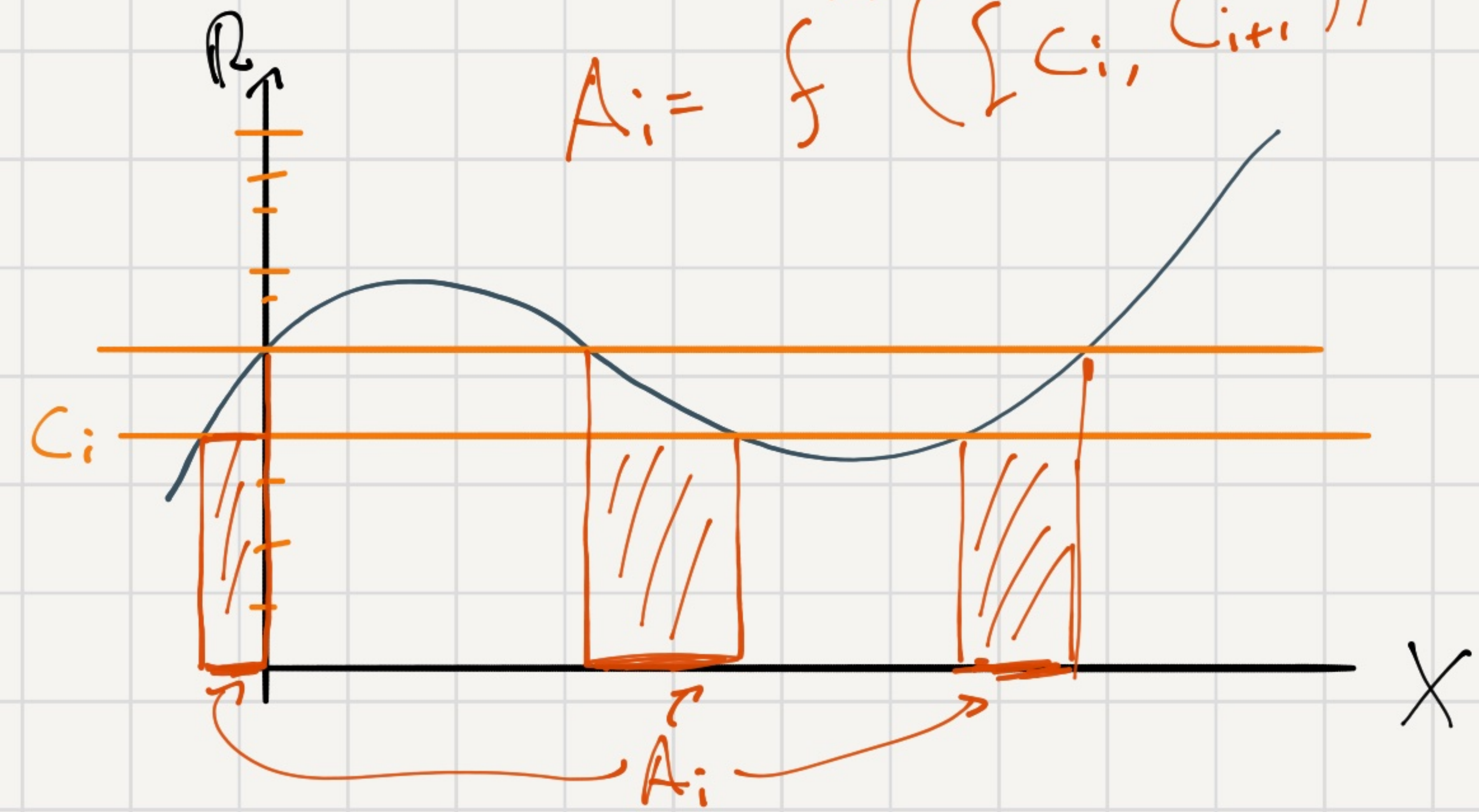
f is called $\underbrace{\mu\text{-integrable}}_{\text{(Lebesgue)}}$ if $\int_X f \, d\mu < \infty$

Riemann vs. Lebesgue

Compare to the Riemann's philosophy



$$A_i = f^{-1}([c_i, c_{i+1}))$$



$$h = \sum c_i \chi_{A_i}$$

One of the two main theorems !?

Theorem of monotone convergence.

$(X, \mathcal{A}, \mu)$

Let $0 \leq f_1 \leq f_2 \leq \cdots$ measurable functions, $f_n : X \to \mathbb{R}$.

Let $f \triangleq \lim_{n\to\infty} f_n \in \overline{\mathbb{R}}_+$. Then,

point wise

Really ok to take $\overline{\mathbb{R}}_+$

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Pointwise convergence $\Rightarrow$ $\int \lim = \lim \int$!?

Cor. $\forall\ 0 \leq f_1 \leq f_2 \leq \cdots$ as above

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

Integrable Functions

Def. Let $(X, \mathcal{A}, \mu)$ be a measure space. Define

$$\mathcal{L}^1(\mu) = \left\{ f : X \to \overline{\mathbb{R}} \text{ measurable} \;\middle|\; \int_X |f| \, d\mu < \infty \right\}$$

Recall f measurable $\Rightarrow$ $|f|$ measurable

Functions in $\mathcal{L}(\mu)$ are called μ-integrable.

Any $f \in \mathcal{L}^1(\mu)$ can be written as $f = f^+ - f^-$ for $f^+, f^- \in S^+$. We define

$$\int_X f \, d\mu \triangleq \int_X f^+ \, d\mu - \int_X f^- \, d\mu$$

note no $\infty - \infty$ issue

Notation.

It is often convenient to "refer to a variable"

$$\int f(x)\, d\mu(x)$$

OR

$$\int f(x)\, \mu(dx)$$

instead of $\int f\, d\mu$

function values rather than the function itself

E.g. $\int x^2\, d\mu(x)$

Complex-valued integrals.

This is inherited by defining for $f : X \to \mathbb{C}$, we define

$$\int f \, d\mu = \int \operatorname{Re} f \, d\mu + i \cdot \int \operatorname{Im} f \, d\mu$$

f is μ-integrable $\iff$ $\operatorname{Re} f$ & $\operatorname{Im} f$ are μ-integrable.

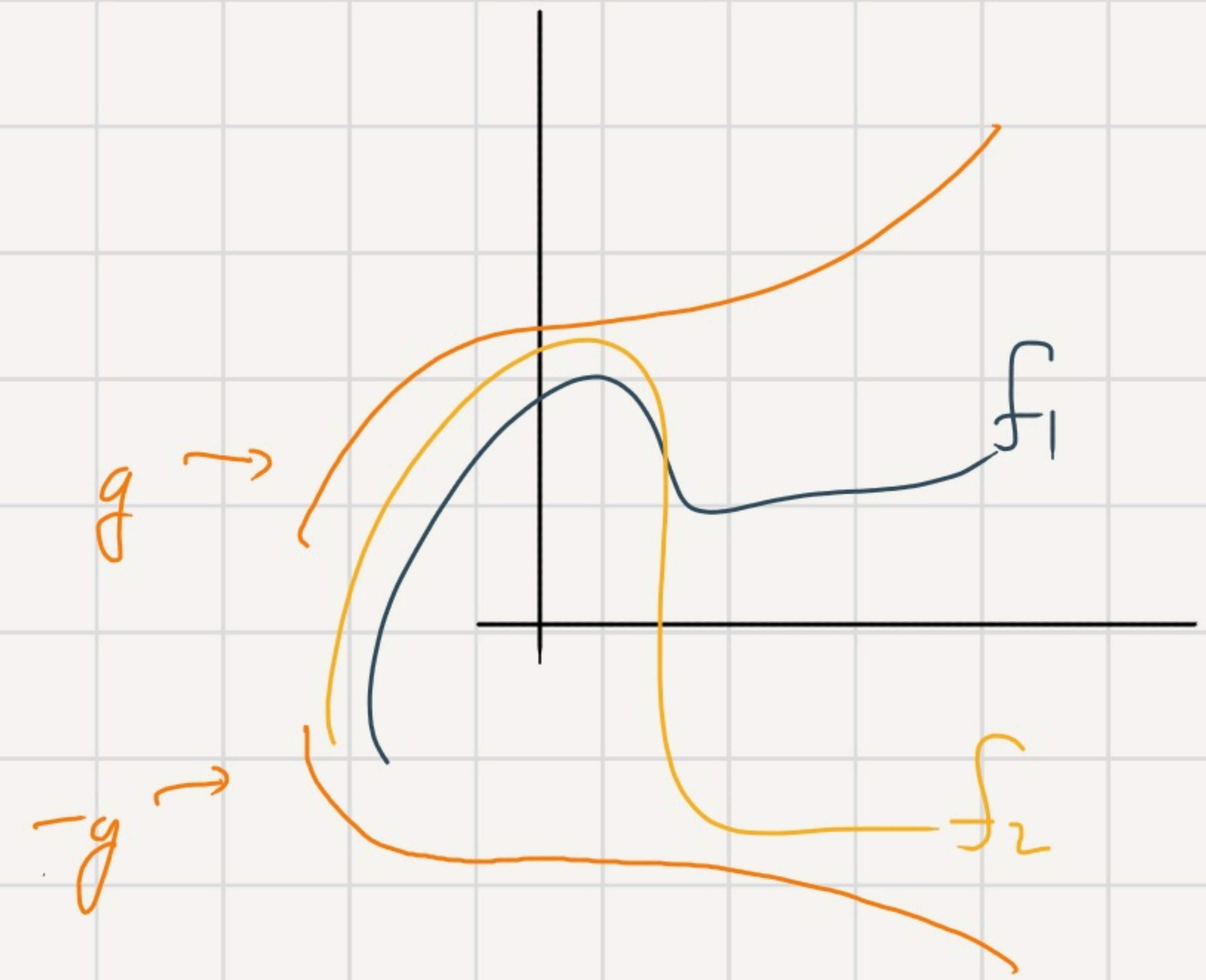
The second main theorem

Theorem of dominant convergence.

Let $f_1, f_2, \ldots$ measurable functions, $f_n : X \to \overline{\mathbb{R}}$.

no need to worry about integrability

Assume $f_n \xrightarrow[a.e]{} f$ pointwise



Let $g \geq 0$ & integrable s.t $\forall n \ |f_n| \leq_{a.e} g$

Then, f & $\forall f_n$ are integrable &

$$\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$$

Integrating against the Dirac measure

Thm. Let X be a set, $p \in X$. Consider the measure space $(X, P(X), \delta_p)$. Let $f \in \mathcal{L}^1(\delta_p)$. Then,

$$\int_X f \, d\delta_p = f(p)$$

pf. Let $g \in S^+$, $g(x) = \sum_i c_i \chi_{A_i}(x)$. Then,

$$\int_X g \, d\delta_p = \sum_i c_i \underbrace{\delta_p(A_i)}_{= \chi_{A_i}(p)} = g(p).$$

For $f \geq 0$ measurable,

$$\Longrightarrow \int_X f \, d\delta_p = \sup \{ g(p) \mid g \leq f, \, g \in S^+ \} = f(p)$$

$\leq$ is obvious,
$\geq$ take $g(x) = \begin{cases} f(p) & x = p \\ 0 & \text{o.w.} \end{cases}$

From here it is easy to prove for any integrable f.