Fiedler Value and Cheeger's Inequality

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Overview

The Fiedler value

The Laplacian of a graph is PSD. Indeed, recall

$$\mathbf{x}^T \mathbf{L} \mathbf{x} = \sum_{uv \in E} (\mathbf{x}(u) - \mathbf{x}(v))^2. \tag{1}$$

If λ is an eigenvalue of $\mathbf{L}$ with (normalized) eigenvector ψ then

$$0 \leq \psi^T \mathbf{L} \psi = \lambda$$

We sort the eigenvalues of the Laplacian from smallest to largest

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Note that $\lambda_1 = 0$ by Equation 1. Alternatively,

$$\mathbf{L1} = (\mathbf{D} - \mathbf{M})\mathbf{1} = \mathbf{0}.$$

The Fiedler value

Lemma

G is connected $\iff \lambda_2 > 0$.

Proof

If G is not connected we can write

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_1 & 0 \\ 0 & \mathbf{L}_2 \end{pmatrix}.$$

Hence $\binom{\mathbf{1}}{\mathbf{0}}$ and $\binom{\mathbf{0}}{\mathbf{1}}$ are two orthogonal eigenvectors of eigenvalue 0.

The Fiedler value

Proof.

Assume now that $\lambda_2 = 0$. We wish to show that G is not connected.

Let ψ be an eigenvector corresponding to λ_2. We have

$$0 = \lambda_2 = \psi^T \mathbf{L} \psi = \sum_{uv \in E} (\psi(u) - \psi(v))^2.$$

From this it follows that ψ is constant on every connected component. But ψ is not constant (as $\psi^T \mathbf{1} = 0$, $\psi \neq \mathbf{0}$) and so G cannot be connected. □

Overview

Isoperimetry and λ_2

Definition

Let $G = (V, E)$ be an undirected graph. For $S \subseteq V$ we define the boundary of S by

$$\partial(S) = \{uv \in E \mid u \in S, v \notin S\}.$$

The isoperimetric ratio of $S \neq \emptyset$ is defined by

$$\theta(S) = \frac{|\partial(S)|}{|S|}.$$

The isoperimetric ratio of G is given by

$$\theta_G = \min_{\emptyset \neq S \subset V} \max(\theta(S), \theta(V \setminus S)) = \min_{0 < |S| \leq \frac{n}{2}} \theta(S).$$

Cheeger's inequality

Clearly, for a d-regular graph G, $\theta_G \leq d$. (in fact, $\theta_G \leq \frac{d}{2} + o(1)$).

We say that G is α-edge expander if $\theta_G \geq \alpha d$

Theorem (Cheeger's inequality)

For every d-regular graph G,

$$\frac{\lambda_2}{2} \leq \theta_G \leq \sqrt{2\lambda_2 d}.$$

Hence, a d-regular graph is $\alpha = \frac{\lambda_2}{2d}$ edge expander.

Namely, a large $\lambda_2 \implies$ good edge expansion.

Overview

Cheeger's inequality - easy direction

Proof

To prove the easy direction we show that for every $S \subseteq V$ with $s = \frac{|S|}{n}$ $(n = |V|)$,

$$\theta(S) \geq \lambda_2(1 - s).$$

By Courant-Fischer,

$$\lambda_2 = \min_{\mathbf{x} \perp \mathbf{1}} \frac{\mathbf{x}^T \mathbf{L} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

Take $\mathbf{x} = \mathbf{1}_S - s\mathbf{1}$. Then,

$$\mathbf{x}^T \mathbf{L} \mathbf{x} = (\mathbf{1}_S - s\mathbf{1})^T \mathbf{L} (\mathbf{1}_S - s\mathbf{1}) = \mathbf{1}_S^T \mathbf{L} \mathbf{1}_S.$$

But

$$\mathbf{1}_S^T \mathbf{L} \mathbf{1}_S = \sum_{uv \in E} (\mathbf{1}_S(u) - \mathbf{1}_S(v))^2 = \partial(S).$$

Cheeger's inequality - easy direction

Proof

Now,

$$\begin{aligned}\mathbf{x}^T\mathbf{x} &= (\mathbf{1}_S - s\mathbf{1})^T(\mathbf{1}_S - s\mathbf{1}) \\ &= \mathbf{1}_S^T\mathbf{1}_S - 2s\mathbf{1}_S^T\mathbf{1} + s^2\mathbf{1}^T\mathbf{1} \\ &= sn - 2s^2n + s^2n \\ &= sn - s^2n \\ &= |S|(1-s).\end{aligned}$$

Combining the above,

$$\lambda_2 \leq \frac{\partial(S)}{|S|(1-s)} \implies \theta(S) = \frac{|\partial(S)|}{|S|} \geq \lambda_2(1-s).$$

Overview

Cheeger's inequality - difficult direction idea

To prove the more difficult direction one uses an eigenvector of the second eigenvalue to embed the graph in the real line.

Figure: 20-vertex path graph embedded into $\mathbb{R}$.

Figure: 20-vertex cycle graph embedded into $\mathbb{R}$.

Graph drawing using a second eigenvector

Figure: Depth-4 complete binary tree embedded into $\mathbb{R}$.

Figure: Third-dumbbell embedded into $\mathbb{R}$.

Cheeger's inequality - difficult direction idea

To prove the more difficult direction one uses an eigenvector of the second eigenvalue to embed the graph in the real line.
Indeed,

$$\lambda_2 = \min_{\substack{\mathbf{x} \perp \mathbf{1} \\ \|\mathbf{x}\|=1}} \mathbf{x}^T \mathbf{L} \mathbf{x} = \min_{\substack{\mathbf{x} \perp \mathbf{1} \\ \|\mathbf{x}\|=1}} \sum_{uv \in E} (\mathbf{x}(u) - \mathbf{x}(v))^2,$$

and so such an eigenvector will tend to embed the vertices in a "cluster" close to each other.

Then one defines a distribution on the reals according to the embedding, and use it to cut the vertex set to two sets.

Overview

Summary

- The larger λ_2 the better is the edge expansion.
- For a d-regular graph, $\lambda_2 = d - \mu_2$. So, equivalently, the smaller μ_2 $(=d\omega_2)$ is the better is the expansion.
- How small can μ_2 be?