Spectral Graph Theory **Winter 2020**

Problem Set 1

Publish Date: October 27, 2020 *Due Date: November 10, 2020*

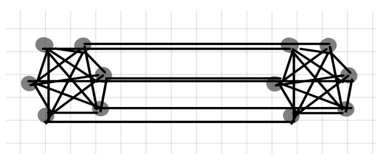
Exercise 1.1 *Let $G = \{[n], E\}$ be a graph and let $\sigma \in S_n$ be a permutation. Define $\sigma(G) = \{[n], \sigma(E)\}$ where $(\sigma(i), \sigma(j)) \in \sigma(E) \iff (i,j) \in E$. Prove or disprove the following statements:*

(a) $Spec(L_G) = Spec(L_{\sigma(G)})$.

(b) v is an eigenvector of L_G iff v is an eigenvector of $L_{\sigma(G)}$.

Exercise 1.2 *Let G_1, G_2 be two graphs on n and m vertices respectively. Prove that $L_{G_1 \times G_2} = L_{G_1} \otimes I_m + I_n \otimes L_{G_2}$ where $\otimes$ is the Kronecker product of the matrices and $\times$ is the graph product defined in class. Denote by $\{\lambda_1, \dots, \lambda_n\}, \{\mu, \dots, \mu_m\}$ the eigenvalues of L_{G_1}, L_{G_2} respectively. Prove that the set $\{\lambda_i + \mu_j \mid i \in [n], j \in [m]\}$ is the set of eigenvalues of $L_{G_1 \times G_2}$.*

Exercise 1.3 *Let G be a graph containing two disjoint cliques on n vertices with a single perfect matching between them. Compute the eigenvalues of L_G and provide an orthogonal basis of eigenvectors. For example:*



Exercise 1.4 *Let $G = ([n], E)$ be a $d-$regular graph. Prove that the second smallest eigenvalue of L_G, satisfies $\lambda_2 \leq \frac{n}{n-1}d$. Prove that this bound is tight, that is, for every n there is a d-regular $G = ([n], E)$ such that $\lambda_2 = \frac{n}{n-1}d$.*

Exercise 1.5 *Let $\{v_1, \dots, v_r\} \subset \mathbb{R}^n$ be vectors such that $\|v_i\|_2 = 1$ for every $i \in [r]$.*

(a) Prove that if $< v_i, v_j >= 0$ for every $i \neq j$ then $\dim(\{v_1, \dots, v_r\}) = r$.

(b) Let A be a symetric matrics with eigenvalues $\lambda_1, \dots, \lambda_t$ prove that $Tr(A^2) = \sum_{i=0}^{t} \lambda_i^2$.

(c) Prove that if $|< v_i, v_j >| \leq \epsilon$ for every $i \neq j$ then $\dim(\{v_1, \dots, v_r\}) \geq \frac{r}{1+(r-1)\epsilon^2}$. Hint: use item (b) (on a suitable matrix) and Cauchy-Schwartz inequality.

Exercise 1.6 *Recall that we proved in class that $P^t L_{R_{2n}} P = 2L_{P_n}$. Where,*

$$P = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in M(\mathbb{R})_{2n \times n},$$

and that if $v = Pu$ is a eigenvector of $L_{R_{2n}}$ satisfying $L_{R_{2n}} v = \lambda v$ then $L_{P_n} u = \lambda u$.

(a) Prove that for every $\lambda \in Spec(L_{R_{2n}})$ of multiplicity 2 there is a corresponding eigenvector $v_\lambda \in Im(P)$.

(b) Compute $Spec(L_{P_n})$ and provide a basis of eigenvectors.