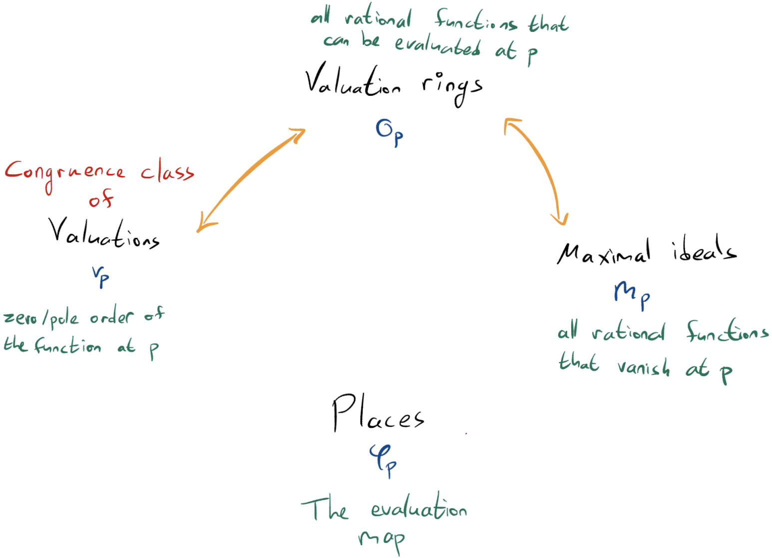
Places

Unit 6

Gil Cohen

November 15, 2024

So far


all rational functions that can be evaluated at p
Valuation rings
$\mathcal{O}_p$
Congruence class of Valuations
v_p
zero/pole order of the function at p
Maximal ideals
m_p
all rational functions that vanish at p
Places
φ_p
The evaluation map

Overview

Adjoining ∞ to a field

Let K be a field. We adjoin to K an element ∞ and extend the operations so that

$$\begin{array}{ll} \forall a \in \mathrm{K} & a \pm \infty = \pm\infty + a = \infty \\ \forall a \in \mathrm{K}^\times & a \cdot \infty = \infty \cdot a = \infty \cdot \infty = \infty \\ \forall a \in \mathrm{K} & \dfrac{a}{\infty} = 0 \\ \forall a \in \mathrm{K}^\times & \dfrac{a}{0} = \infty. \end{array}$$

Moreover, the expressions

$$\infty \pm \infty \quad 0 \cdot \infty \quad \infty \cdot 0 \quad \frac{0}{0} \quad \frac{\infty}{\infty}$$

are undefined.

You should think of a as the result of an evaluation and interpret ∞ as evaluation was impossible due to a pole.

Places

In the definition, think of F as a field of functions whereas K is the field of possible evaluation outcomes at some fixed point.

Definition 1 (Place)

Let F, K be fields. A map

$$\varphi : F \to K \cup \{\infty\}$$

is called a place if

1. $\varphi(1) = 1$
2. $\varphi(a + b) = \varphi(a) + \varphi(b)$ whenever at least one of $\varphi(a), \varphi(b)$ is not ∞ (or, if you prefer, $\{\varphi(a), \varphi(b)\} \neq \{\infty\}$.)
3. $\varphi(ab) = \varphi(a)\varphi(b)$ whenever $\{\varphi(a), \varphi(b)\} \neq \{0, \infty\}$.

Overview

Places - examples

Example

For a prime p let $\mathbb{F}_p$ be the field of size p. Recall that $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$. Denote by $\psi : \mathbb{Z} \to \mathbb{F}_p$ the projection map $\psi(z) = z + p\mathbb{Z}$.

We extend the ring homomorphism ψ to a place

$$\varphi : \mathbb{Q} \to \mathbb{F}_p \cup \{\infty\}$$

as follows:

Given $q \in \mathbb{Q}$ write $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ coprime.

Define,

$$\varphi(q) = \begin{cases} \frac{\psi(a)}{\psi(b)}, & p \text{ does not divide } b; \\ \infty, & \text{otherwise.} \end{cases}$$

I leave it for you as an exercise to show that φ is indeed a place.

Places - examples

Example

Let E be a field and $p(x) \in \mathsf{E}[x]$ irreducible. Let

$$\psi : \mathsf{E}[x] \to \mathsf{L} = \mathsf{E}[x] \big/ \langle p(x) \rangle$$

be the projection map $\psi(f(x)) = f(x) + \langle p(x) \rangle$.

We extend the ring homomorphism ψ to a place

$$\varphi : \mathsf{E}(x) \to \mathsf{L} \cup \{\infty\}$$

as follows: Given $f(x) \in \mathsf{E}(x)$ write $f(x) = \frac{a(x)}{b(x)}$ with $a(x), b(x) \in \mathsf{E}[x]$ coprime, and define

$$\varphi(f(x)) = \begin{cases} \frac{\psi(a(x))}{\psi(b(x))}, & p(x) \text{ does not divide } b(x); \\ \infty, & \text{otherwise.} \end{cases}$$

Places - examples

Example

Recall that

$$\psi : \mathsf{E}[x] \to \mathsf{L} = \mathsf{E}[x] \Big/ \langle p(x) \rangle$$

is the projection map $\psi(f(x)) = f(x) + \langle p(x) \rangle$.

In the special case $p(x) = x - \alpha$ we can think of ψ as "evaluating at α" since then

$$\psi : \mathsf{E}[x] \to \mathsf{L} = \mathsf{E}[x] \Big/ \langle x - \alpha \rangle \cong \mathsf{E},$$

and for every $f(x) \in \mathsf{E}[x]$,

$$\psi(f(x)) = f(x) + \langle x - \alpha \rangle = f(\alpha) + \langle x - \alpha \rangle.$$

Moreover, note that $f(\alpha)$ is the only representative in the coset $\psi(f(x))$ that is an element of E.

Places - examples

Example

$$\psi : \mathsf{E}[x] \to \mathsf{L} = \mathsf{E}[x] \Big/ \langle x - \alpha \rangle \cong \mathsf{E}.$$

Now, $\varphi : \mathsf{E}(x) \to \mathsf{L} \cup \{\infty\}$ is given by

$$\varphi(f(x)) = \begin{cases} \frac{\psi(a(x))}{\psi(b(x))}, & x - \alpha \text{ does not divide } b(x); \\ \infty, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{a(\alpha) + \langle x - \alpha \rangle}{b(\alpha) + \langle x - \alpha \rangle}, & b(\alpha) \neq 0; \\ \infty, & \text{otherwise.} \end{cases}$$

Under the identification of L with E as given by

$$g(x) + \langle x - \alpha \rangle \quad \longleftrightarrow \quad g(\alpha),$$

we can write

$$\varphi(f(x)) = \begin{cases} \frac{a(\alpha)}{b(\alpha)}, & b(\alpha) \neq 0; \\ \infty, & \text{otherwise.} \end{cases}$$

Overview

Trivial places

A quick reminder re field homomorphisms.

A field homomorphism $\psi : \mathsf{F} \to \mathsf{K}$ is always a monomorphism. Indeed, as ψ is a ring homomorphism, $\ker \psi$ is an ideal of F. The only ideals of F are 0 and F. But $\varphi(1) = 1$ and so $1 \notin \ker \psi$. Thus, $\ker \psi = 0$, implying ψ is a monomorphism.

By the above remark, ψ is thought of as a field embedding $\mathsf{F} \hookrightarrow \mathsf{K}$. Namely, we can identify F with $\varphi(\mathsf{F}) \subseteq \mathsf{K}$.

Definition 2

A place $\varphi : \mathsf{F} \to \mathsf{K} \cup \{\infty\}$ is called trivial if $\varphi(a) \neq \infty$ for all $a \in \mathsf{F}$.

By the above reminder, a trivial place is a field embedding, and vice versa.

Equivalent places

Definition 3

Two places $\varphi : \mathsf{F} \to \mathsf{K} \cup \{\infty\}$, $\varphi' : \mathsf{F} \to \mathsf{K}' \cup \{\infty\}$ are equivalent if $\forall a \in \mathsf{F}$,

$$\varphi(a) \neq \infty \iff \varphi'(a) \neq \infty.$$

We note that a trivial place $\varphi : \mathsf{F} \to \mathsf{K} \cup \{\infty\}$ is equivalent to the identity field isomorphism $\mathrm{id}_\mathsf{F} : \mathsf{F} \to \mathsf{F}$.

For distinct $\alpha, \beta \in \mathsf{K}$, the places $\varphi_\alpha, \varphi_\beta$ of $\mathsf{K}(x)$ that correspond to $x - \alpha$ and $x - \beta$ are not equivalent. Indeed,

$$\varphi_\alpha\left(\frac{1}{x-\alpha}\right) = \infty \qquad \varphi_\beta\left(\frac{1}{x-\alpha}\right) = \frac{1}{\beta-\alpha}.$$

So, distinct points in the field K give rise to distinct places of $\mathsf{K}(x)$.

Same holds for any two distinct irreducible polynomials.

Overview

Basic properties of places

Claim 4

Let $\varphi : \mathsf{F} \to \mathsf{K} \cup \{\infty\}$ be a place. Then,

1. $\varphi(0) = 0$.
2. $\varphi(-a) = -\varphi(a)$. In particular, $\varphi(a) = \infty \iff \varphi(-a) = \infty$.
3. If $\varphi(a), \varphi(b) \neq \infty$ then $\varphi(a+b) \neq \infty$.
4. $\varphi(a) = \infty \iff \varphi(a^{-1}) = 0$.

Proof.

As for Item (1),

$$\varphi(1) = \varphi(1+0) = \varphi(1) + \varphi(0) \quad \Rightarrow \quad \varphi(0) = 0.$$

□

Basic properties of places

Proof.

$$\varphi(1) = \varphi(1+0) = \varphi(1) + \varphi(0) \quad \Rightarrow \quad \varphi(0) = 0.$$

In the above derivation there are two subtleties:

1. $\varphi(1) = 1 \neq \infty$ and so the second equality holds.
2. The implication follows by "canceling $\varphi(1)$". However, we should be careful. $\varphi(1) = 1$ and so we need to show that

$$1 = 1 + \varphi(0) \quad \Rightarrow \quad \varphi(0) = 0.$$

If $\varphi(0) = \infty$ then $1 = 1 + \infty$ - a contradiction. Thus, $\varphi(0) \neq \infty$ and so the entire expression is in the field K which allows us to substract 1 and deduce $\varphi(0) = 0$.

□

Basic properties of places

Proof.

As for the second item, if $\varphi(a) \neq \infty$ then

$$0 = \varphi(0) = \varphi(a + (-a)) = \varphi(a) + \varphi(-a).$$

Now, if $\varphi(-a) = \infty$ then we would get

$$0 = \varphi(a) + \infty$$

a contradiction. Thus, $\varphi(-a) \in \mathsf{K}$, implying $\varphi(-a) = -\varphi(a)$.

If on the other hand $\varphi(a) = \infty$ and $\varphi(-a) \neq \infty$ then the RHS is $\infty \neq 0$.

Basic properties of places

Proof.

To prove the third item, we recall that

$$\varphi(a) \neq \infty \iff a \in \mathsf{K}.$$

Thus, our assumption implies that $a, b \in \mathsf{K}$, and so $a + b \in \mathsf{K}$. This then implies $\varphi(a+b) \neq \infty$.

Basic properties of places

Proof.

As for the fourth item,

$$1 = \varphi(1) = \varphi(aa^{-1}).$$

If $\varphi(a) = \infty$ and $\varphi(a^{-1}) \neq 0$ then

$$1 = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1}) = \infty.$$

Hence, $\varphi(a) = \infty \implies \varphi(a^{-1}) = 0$.

On the other hand, if $\varphi(a^{-1}) = 0$ and $\varphi(a) = c \neq \infty$ then

$$1 = \varphi(a)\varphi(a^{-1}) = c \cdot 0 = 0$$

which again is a contradiction.

Overview

The residue field

Claim 5

Let $\varphi : \mathsf{F} \to \mathsf{K} \cup \{\infty\}$ be a place. Then, $\bar{\mathsf{F}} = \varphi(\mathsf{F}) \setminus \{\infty\}$ is a subfield of K.

Proof.

It easy to see that $\bar{\mathsf{F}}$ is closed under addition and multiplication. E.g., if $\alpha, \beta \in \bar{\mathsf{F}}$ then $\exists a, b \in \mathsf{F}$ s.t. $\alpha = \varphi(a)$, $\beta = \varphi(b)$. Thus,

$$\alpha + \beta = \varphi(a) + \varphi(b) = \varphi(a + b),$$

and so $\alpha + \beta \in \bar{\mathsf{F}}$.

Similarly, $\bar{\mathsf{F}}$ is closed under negation.

The residue field

Proof.

It is left to show $\bar{F} \setminus \{0\}$ is closed under multiplicative inverse.

Let $\alpha \in \bar{F} \setminus \{0\}$ and let $a \in F$ s.t. $\varphi(a) = \alpha$. Note that $\varphi(a^{-1}) \neq \infty$ as otherwise, Claim 4 would imply $\varphi(a) = 0$.

Thus,

$$\alpha^{-1} = \varphi(a)^{-1} = \varphi(a^{-1}) \in \bar{F},$$

where the last equality follows since

$$1 = \varphi(1) = \varphi(a \cdot a^{-1}) = \varphi(a)\varphi(a^{-1}),$$

where for the last equality we used the fact that $\varphi(a) \neq 0$.

Lastly, we recall that $\varphi(1) = 1$ and so $1 \in \bar{F}$. □

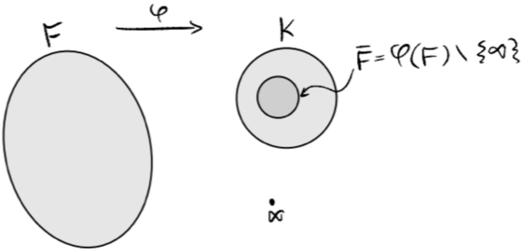
The residue field

Definition 6 (The residue field)

Let $\varphi : F \to K \cup \{\infty\}$ be a place. The field,

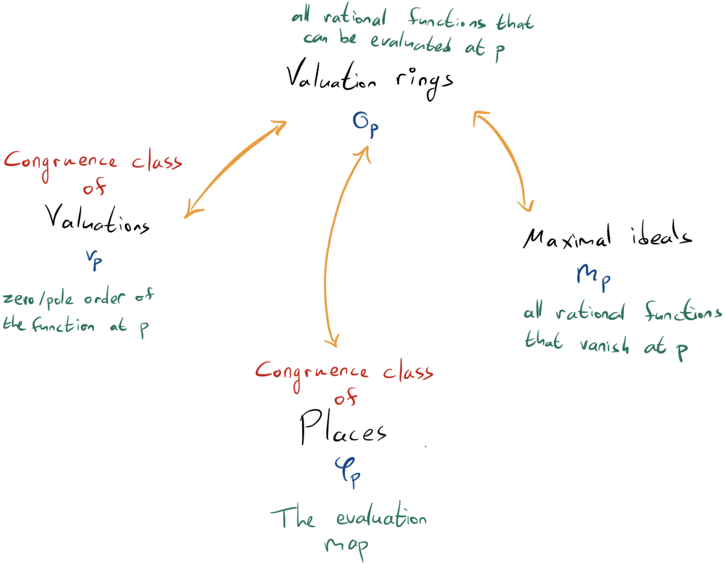
$$\bar{F} = \varphi(F) \setminus \{\infty\}$$

is called the residue field of φ.



Overview

Places and valuation rings



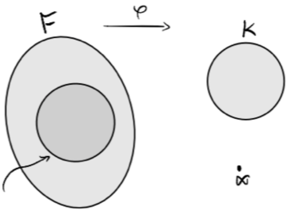
Places and valuation rings

Claim 7

Let $\varphi : F \to K \cup \{\infty\}$ be a place. Then,

$$\mathcal{O}_\varphi = \{a \in F \mid \varphi(a) \neq \infty\}$$

is a valuation ring with $\operatorname{Frac} \mathcal{O}_\varphi = F$.



$$\mathcal{O}_\varphi = \{a \in F \mid \varphi(a) \neq \infty\}$$
$$= \{a \in F \mid \varphi(a) \in K\}$$
$$= \{a \in F \mid \varphi(a) \in \overline{F}\}$$

Places and valuation rings

Proof.

First, $\varphi(1) = 1$ by the definition of a place and so $1 \in \mathcal{O}_\varphi$.

To prove that $\mathcal{O}_\varphi$ is closed under addition, we use Claim 4 to get

$$\begin{aligned} a, b \in \mathcal{O}_\varphi &\iff \varphi(a), \varphi(b) \neq \infty \\ &\implies \varphi(a+b) \neq \infty \\ &\iff a + b \in \mathcal{O}_\varphi. \end{aligned}$$

That $\mathcal{O}_\varphi$ is closed under multiplication is proven by a similar argument. Thus, $\mathcal{O}_\varphi$ is a subring of F.

We turn to prove that $\mathcal{O}_\varphi$ is a valuation ring with field of fractions F.

Take $a \in \mathrm{F}^\times$ with $a \notin \mathcal{O}_\varphi$. Then, $\varphi(a) = \infty$ and so, by Claim 4, $\varphi(a^{-1}) = 0 \neq \infty$. Thus, $a^{-1} \in \mathcal{O}_\varphi$. We further conclude that $\operatorname{Frac} \mathcal{O}_\varphi = \mathrm{F}$.

Places and valuation rings

Claim 8

Let $\varphi : \mathsf{F} \to \mathsf{K} \cup \{\infty\}$ be a place. Then,

$$\begin{aligned}\mathcal{O}_\varphi^\times &= \{a \in \mathsf{F} \mid \varphi(a) \notin \{0, \infty\}\} \\ &= \{a \in \mathcal{O}_\varphi \mid \varphi(a) \neq 0\} \\ &= \mathcal{O}_\varphi \setminus \ker \varphi.\end{aligned}$$

Proof.

By Claim 4,

$$\begin{aligned} a \in \mathcal{O}_\varphi^\times \quad &\iff \quad a, a^{-1} \in \mathcal{O}_\varphi \\ &\iff \quad \varphi(a), \varphi(a^{-1}) \neq \infty \\ &\iff \quad \varphi(a) \notin \{0, \infty\}. \end{aligned}$$

□

Claim 9

Let $\varphi, \varphi' : \mathsf{F} \to \mathsf{K} \cup \{\infty\}$ be equivalent places. Then,

$$\mathcal{O}_\varphi = \mathcal{O}_{\varphi'}.$$

Proof.

This is straightforward by definition. Indeed,

$$\begin{aligned}\mathcal{O}_\varphi &= \{a \in \mathsf{F} \mid \varphi(a) \neq \infty\} \\ &= \{a \in \mathsf{F} \mid \varphi'(a) \neq \infty\} \\ &= \mathcal{O}_{\varphi'}.\end{aligned}$$

□

Let K, F be fields and let $\varphi : \mathsf{F} \to \mathsf{K} \cup \{\infty\}$ be a place. We denote by $[\varphi]$ the equivalent class of φ.

Places and valuation rings

Theorem 10

The map

$$[\varphi] \mapsto \mathcal{O}_\varphi$$

is a bijection between the congruence classes of places of F and valuation rings with fraction field F.

Proof.

First, by Claim 9, the map is well-defined.

The one to one property is obvious. We prove that the mapping is onto.

Let R be a valuation ring with $\operatorname{Frac} R = F$. Let $\mathfrak{m}$ be R's maximal ideal and let $K = R/\mathfrak{m}$.

We extend the projection map $\psi : R \to K$ to F by setting $\psi(a) = \infty$ for all $a \in F \setminus R$. We turn to show that ψ is a place.

Places and valuation rings

Proof.

Let $a, b \in \mathrm{F}$. We wish to show that

$$\psi(a+b) = \psi(a) + \psi(b)$$

whenever (at least) one of $\psi(a), \psi(b)$ is not ∞.

Case 1. $\psi(a), \psi(b) \neq \infty$ immediately follows.

Case 2. $\psi(a) \neq \infty$ and $\psi(b) = \infty$. Then,

$$\psi(a) + \psi(b) = \psi(a) + \infty = \infty.$$

On the other hand, $a + b \notin \mathrm{R}$ as otherwise $b = (a+b) - a \in \mathrm{R}$, and so $\psi(a+b) = \infty$.

Places and valuation rings

Proof.

We turn to show that $\psi(ab) = \psi(a)\psi(b)$ when $\{\psi(a), \psi(b)\} \neq \{0, \infty\}$.

Case 1. $\psi(a), \psi(b) \neq \infty$ immediately follows.

Case 2. $\psi(a) = \infty$ and $\psi(b) \neq 0$. Then, $a \notin \mathrm{R}$. Further,

$$\psi(b^{-1}) \neq \infty$$

as otherwise $\psi(b) = 0$. Thus, $b^{-1} \in \mathrm{R}$. Now, if $\psi(ab) \neq \infty$ then $ab \in \mathrm{R}$ and so

$$a = (ab)b^{-1} \in \mathrm{R},$$

in contradiction to $a \notin \mathrm{R}$. Thus,

$$\psi(ab) = \infty = \psi(a)\psi(b).$$

Overview

Example

Recall the example from the previous unit. Let $\mathsf{K} = \mathbb{F}_q$, let

$$f(x,y) = y^2 - x^3 + x \in K[x,y],$$

and consider the domain

$$C_f = \mathsf{K}[x,y] \Big/ \langle f(x,y) \rangle$$

whose field of fractions is denoted by $\mathsf{K}_f = \text{Frac } C_f$. We proved that

$$\mathcal{O}_o = \left\{ \frac{a(x)}{b(x)} + y\frac{c(x)}{d(x)} \;\middle|\; b(0), d(0) \neq 0 \right\},$$

with the understanding that $a(T), b(T) \in \mathsf{K}[T]$ are coprime and so are $c(T), d(T) \in \mathsf{K}[T]$. Moreover,

$$\mathfrak{m}_o = \left\{ \frac{a(x)}{b(x)} + y\frac{c(x)}{d(x)} \;\middle|\; b(0), d(0) \neq 0 \text{ and } a(0) = 0 \right\}.$$

Example

$$\mathcal{O}_o = \left\{ \frac{a(x)}{b(x)} + y\frac{c(x)}{d(x)} \;\middle|\; b(0), d(0) \neq 0 \right\},$$
$$\mathfrak{m}_o = \left\{ \frac{a(x)}{b(x)} + y\frac{c(x)}{d(x)} \;\middle|\; b(0), d(0) \neq 0 \text{ and } a(0) = 0 \right\}.$$

We claim that $\mathcal{O}_o \big/ \mathfrak{m}_o \cong \mathsf{K}$. Indeed, consider the ring homomorphism

$$\psi : \mathcal{O}_o \to \mathsf{K}$$
$$\frac{a(x)}{b(x)} + y\frac{c(x)}{d(x)} \;\mapsto\; \frac{a(0)}{b(0)}.$$

ψ is well-defined as $b(0) \neq 0$ for every element of $\mathcal{O}_o$. Clearly, $\ker \psi = \mathfrak{m}_o$, and so $\mathcal{O}_o \big/ \mathfrak{m}_o \cong \mathsf{K}$ by the first isomorphism theorem.

Example

Following the proof of Theorem 10, we extend the projection map

$$\psi : \mathcal{O}_\mathfrak{o} \to \mathsf{K}$$

to

$$\varphi_\mathfrak{o} : \mathsf{K}_f \to \mathsf{K}$$

by setting $\varphi_\mathfrak{o}(a) = \infty$ for all $a \in \mathsf{K}_f \setminus \mathcal{O}_\mathfrak{o}$.

Thus,

$$\varphi_\mathfrak{o}\left(\frac{a(x)}{b(x)} + y\frac{c(x)}{d(x)}\right) = \begin{cases} \frac{a(0)}{b(0)}, & b(0) \neq 0 \text{ and } d(0) \neq 0; \\ \infty, & \text{otherwise.} \end{cases}$$