The Riemann-Roch Theorem and Its Consequences

Unit 15

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Overview

The Riemann-Roch Theorem

Without further ado

Theorem 1 (Riemann-Roch)

Let F/K *be a function field with genus* g. *Let* $\mathfrak{c}$ *be a canonical divisor of* F/K. *Then, for every divisor* $\mathfrak{a}$

$$\dim \mathfrak{a} = \deg \mathfrak{a} + 1 - g + \dim(\mathfrak{c} - \mathfrak{a}).$$

Proof.

Recall that

$$\delta(\mathfrak{a}) = \dim_K \Omega(\mathfrak{a}) = \dim_K \mathbb{A} \Big/ (\Lambda(\mathfrak{a}) + F) = g - 1 - (\deg \mathfrak{a} - \dim \mathfrak{a}).$$

Thus, it suffices to prove that

$$\delta(\mathfrak{a}) = \dim(\mathfrak{c} - \mathfrak{a}).$$

The Riemann-Roch Theorem

Proof.

Let $0 \neq \omega \in \Omega$ s.t. $\mathfrak{c} = (\omega)$. Consider the K-linear homomorphism

$$T : \mathsf{F} \to \Omega$$
$$x \mapsto x\omega$$

We have that

$$\begin{aligned} x \in \mathcal{L}(\mathfrak{c} - \mathfrak{a}) \quad &\iff \quad (x) + (\omega) - \mathfrak{a} \geq 0 \\ &\iff \quad (x\omega) \geq \mathfrak{a} \\ &\iff \quad x\omega \in \Omega(\mathfrak{a}). \end{aligned}$$

So T embeds $\mathcal{L}(\mathfrak{c} - \mathfrak{a})$ in $\Omega(\mathfrak{a})$. Since every element in $\Omega(\mathfrak{a})$ is of the form $x\omega$ for some $x \in \mathsf{F}$, we have equality in dimensions. Thus,

$$\delta(\mathfrak{a}) = \dim_{\mathsf{K}} \Omega(\mathfrak{a}) = \dim_{\mathsf{K}} \mathcal{L}(\mathfrak{c} - \mathfrak{a}) = \dim(\mathfrak{c} - \mathfrak{a}).$$

□

Overview

Degree and dimension of canonical divisors

Corollary 2

Let $\mathfrak{c}$ *be a canonical divisor. Then,*

$$\dim \mathfrak{c} = g,$$
$$\deg \mathfrak{c} = 2g - 2.$$

Proof.

By Riemann-Roch,

$$\dim \mathfrak{a} = \deg \mathfrak{a} + 1 - g + \dim(\mathfrak{c} - \mathfrak{a}).$$

Setting $\mathfrak{a} = 0$ we get

$$1 = \dim 0 = \deg 0 + 1 - g + \dim(\mathfrak{c} - 0) = 1 - g + \dim \mathfrak{c},$$

and so $\dim \mathfrak{c} = g$. Again by Riemann-Roch, now applied with $\mathfrak{a} = \mathfrak{c}$,

$$g = \dim \mathfrak{c} = \deg \mathfrak{c} + 1 - g + \dim(\mathfrak{c} - \mathfrak{c}) \quad \Longrightarrow \quad \deg \mathfrak{c} = 2g - 2.$$

Degree zero divisors

Claim 3

1. If $\mathfrak{a}$ is principal then $\dim \mathfrak{a} = 1$ and $\deg \mathfrak{a} = 0$.
2. If $\deg \mathfrak{a} = 0$ and $\mathfrak{a}$ is not principal then $\dim \mathfrak{a} = 0$.

Proof.

For the first item, it suffices to consider $\mathfrak{a} = 0$ as

$$\dim(\mathfrak{a} + (x)) = \dim \mathfrak{a},$$
$$\deg(\mathfrak{a} + (x)) = \deg \mathfrak{a}.$$

The proof follows as $\dim 0 = 1$, $\deg 0 = 0$.

As for Item (2), assume that $\dim \mathfrak{a} > 0$. Then $(x) + \mathfrak{a} \geq 0$ for some $x \neq 0$. But

$$\deg((x) + \mathfrak{a}) = \deg \mathfrak{a} = 0,$$

and so $(x) + \mathfrak{a} = 0$, namely, $\mathfrak{a} = (x^{-1})$ is principal.

Degree $2g - 2$ divisors

Riemann-Roch: $\dim \mathfrak{a} = \deg \mathfrak{a} + 1 - g + \dim(\mathfrak{c} - \mathfrak{a})$.

Corollary 4 (Degree $2g - 2$ divisors)

If $\deg \mathfrak{a} = 2g - 2$ *and* $\mathfrak{a}$ *is not canonical then* $\dim \mathfrak{a} = g - 1$.

Proof.

By Riemann-Roch with "$\mathfrak{a} = \mathfrak{c} - \mathfrak{a}$", and using that $\deg \mathfrak{c} = 2g - 2$,

$$\begin{aligned} \dim(\mathfrak{c} - \mathfrak{a}) &= \deg(\mathfrak{c} - \mathfrak{a}) + 1 - g + \dim(\mathfrak{c} - (\mathfrak{c} - \mathfrak{a})) \\ &= 1 - g + \dim \mathfrak{a}. \end{aligned}$$

Note that $\mathfrak{c} - \mathfrak{a}$ is not principal as otherwise $\mathfrak{a}$ would be canonical. Indeed,

$$\mathfrak{c} - \mathfrak{a} = (x) \quad \Longrightarrow \quad \mathfrak{a} = \mathfrak{c} - (x) \in \mathcal{W}.$$

By Corollary 2, $\deg(\mathfrak{c} - \mathfrak{a}) = 0$. Therefore, Claim 3 implies $\dim(\mathfrak{c} - \mathfrak{a}) = 0$. Thus, $\dim \mathfrak{a} = g - 1$.

Characterization of canonical divisors

We obtain a characterization of canonical divisors in terms of dimension and degree.

Lemma 5

$$\begin{aligned} \mathfrak{a} \in \mathcal{W} \quad &\iff \quad \deg \mathfrak{a} = 2g - 2 \ \textit{and} \ \dim \mathfrak{a} = g \\ &\iff \quad \deg \mathfrak{a} = 2g - 2 \ \textit{and} \ \dim \mathfrak{a} \neq g - 1. \end{aligned}$$

Proof.

By Corollary 2, for $\mathfrak{a} \in \mathcal{W}$ we have $\deg \mathfrak{a} = 2g - 2$ and $\dim \mathfrak{a} = g$.

By Corollary 4, if $\deg \mathfrak{a} = 2g - 2$ and $\dim \mathfrak{a} \neq g - 1$ then $\mathfrak{a} \in \mathcal{W}$.

Negative degree divisors

Claim 6

If $\deg \mathfrak{a} < 0$ then $\dim \mathfrak{a} = 0$.

Proof.

If $\dim \mathfrak{a} > 0$ then $(x) + \mathfrak{a} \geq 0$ for some $x \neq 0$. But,

$$\deg((x) + \mathfrak{a}) = \deg \mathfrak{a} < 0,$$

which is a contradiction to $(x) + \mathfrak{a} \geq 0$. $\square$

Degree $> 2g - 2$ divisors

Riemann-Roch: $\dim \mathfrak{a} = \deg \mathfrak{a} + 1 - g + \dim(\mathfrak{c} - \mathfrak{a})$.

Corollary 7 (Degree $> 2g - 2$ divisors)

If $\deg \mathfrak{a} > 2g - 2$ *then* $\dim \mathfrak{a} = \deg \mathfrak{a} - g + 1$.

Proof.

By Riemann-Roch with "$\mathfrak{a} = \mathfrak{c} - \mathfrak{a}$"

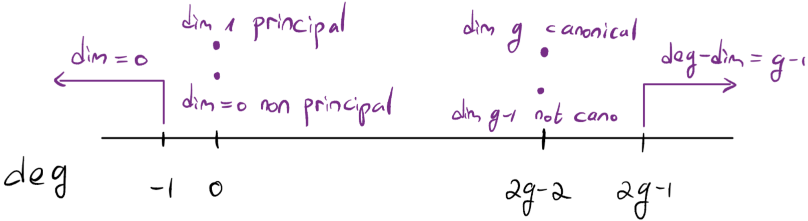
$$\dim(\mathfrak{c} - \mathfrak{a}) = \deg(\mathfrak{c} - \mathfrak{a}) + 1 - g + \dim(\mathfrak{c} - (\mathfrak{c} - \mathfrak{a})).$$

By Corollary 2, $\deg \mathfrak{c} = 2g - 2$ and so $\deg(\mathfrak{c} - \mathfrak{a}) < 0$. Thus, $\dim(\mathfrak{c} - \mathfrak{a}) = 0$ and so

$$\begin{aligned} 0 &= \deg \mathfrak{c} - \deg \mathfrak{a} + 1 - g + \dim \mathfrak{a} \\ &= g - 1 + \dim \mathfrak{a} - \deg \mathfrak{a}, \end{aligned}$$

and the proof follows. $\square$

Summary



dim 1 principal
dim g canonical
dim = 0
deg−dim = g−1
dim = 0 non principal
dim g−1 not cano
deg
−1
0
2g−2
2g−1

Overview

Canonical divisors of the rational function field

By Lemma 5,

$$\mathfrak{a} \in \mathcal{W} \quad \Longleftrightarrow \quad \deg \mathfrak{a} = 2g - 2 \text{ and } \dim \mathfrak{a} = g.$$

We saw that the genus of $\mathsf{K}(x)/\mathsf{K}$ is 0, and so

$$\mathfrak{a} \in \mathcal{W} \quad \Longleftrightarrow \quad \deg \mathfrak{a} = -2 \text{ and } \dim \mathfrak{a} = 0.$$

Note that

$$\deg(-2\mathfrak{p}_\infty) = -2,$$
$$\dim(-2\mathfrak{p}_\infty) = 0,$$

and so $-2\mathfrak{p}_\infty$ is a canonical divisor of the rational function field.

Canonical divisors of the rational function field

Lets try to give some vague informal explanation as to why $-2\mathfrak{p}_\infty$ is a canonical divisor of $\mathsf{K}(x)/\mathsf{K}$.

If we wish to understand dx "at infinity" we can consider $y = \frac{1}{x}$ and then

$$\frac{dx}{dy} = -\frac{1}{y^2} \quad \Longrightarrow \quad dx = -\frac{1}{y^2} \cdot dy.$$

Therefore, we would like to say that dx has a pole of order 2 at $y = 0$, namely, at "$x = \infty$".

Overview

The strong approximation theorem

Theorem 8

Let F/K *be a function field. Let* $S \subseteq \mathbb{P}$ *be finite and* $\mathfrak{q} \in \mathbb{P} \setminus S$. *For every* $\mathfrak{p} \in \mathbb{P}$ *let* $x_{\mathfrak{p}} \in \mathsf{F}$ *and* $m_{\mathfrak{p}} \in \mathbb{Z}$. *Then,* $\exists x \in \mathsf{F}$ *satisfying*

$$\begin{aligned} \forall \mathfrak{p} \in S \quad & \upsilon_{\mathfrak{p}}(x - x_{\mathfrak{p}}) = m_{\mathfrak{p}} \\ \forall \mathfrak{p} \notin \mathsf{S} \cup \{\mathfrak{q}\} \quad & \upsilon_{\mathfrak{p}}(x) \geq 0. \end{aligned}$$

Proof.

We will prove a weaker version in which we get an inequality for all $\mathfrak{p} \in S$ and leave it for you to obtain equality (similarly to the way we did it in the weak approximation theorem). Denote

$$\mathfrak{a} = m\mathfrak{q} - \sum_{\mathfrak{p} \in S} m_{\mathfrak{p}} \mathfrak{p},$$

for m sufficiently large so that

$$\deg \mathfrak{a} > 2g - 2.$$

The strong approximation theorem

Proof.

We took

$$\mathfrak{a} = m\mathfrak{q} - \sum_{\mathfrak{p}\in S} m_{\mathfrak{p}}\mathfrak{p},$$

for m sufficiently large so that $\deg \mathfrak{a} > 2g - 2$. Thus,

$$\deg \mathfrak{a} - \dim \mathfrak{a} = g - 1,$$

and so

$$\mathbb{A} = \Lambda(\mathfrak{a}) + \mathrm{F}.$$

Define the adele α with

$$\alpha_{\mathfrak{p}} = \begin{cases} x_{\mathfrak{p}}, & \mathfrak{p} \in \mathrm{S}, \\ 0, & \text{otherwise.} \end{cases}$$

The strong approximation theorem

Proof.

Recall that $\mathbb{A} = \Lambda(\mathfrak{a}) + \mathsf{F}$, and

$$\mathfrak{a} = m\mathfrak{q} - \sum_{\mathfrak{p}\in S} m_{\mathfrak{p}}\mathfrak{p},$$

$$\alpha_{\mathfrak{p}} = \begin{cases} x_{\mathfrak{p}}, & \mathfrak{p} \in \mathsf{S}, \\ 0, & \text{otherwise.} \end{cases}$$

So $\exists x \in \mathsf{F}$ s.t. $x - \alpha \in \Lambda(\mathfrak{a})$. Thus,

$$\forall \mathfrak{p} \in \mathbb{P} \quad \upsilon_{\mathfrak{p}}(x - \alpha) + \upsilon_{\mathfrak{p}}(\mathfrak{a}) \geq 0.$$

So $\forall \mathfrak{p} \in S$,

$$\upsilon_{\mathfrak{p}}(x - x_{\mathfrak{p}}) - m_{\mathfrak{p}} \geq 0,$$

and for $\mathfrak{p} \notin S \cup \{\mathfrak{q}\}$,

$$\upsilon_{\mathfrak{p}}(x) \geq 0.$$

□