Analytic insights into the Zig-Zag product and its friends

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joint with

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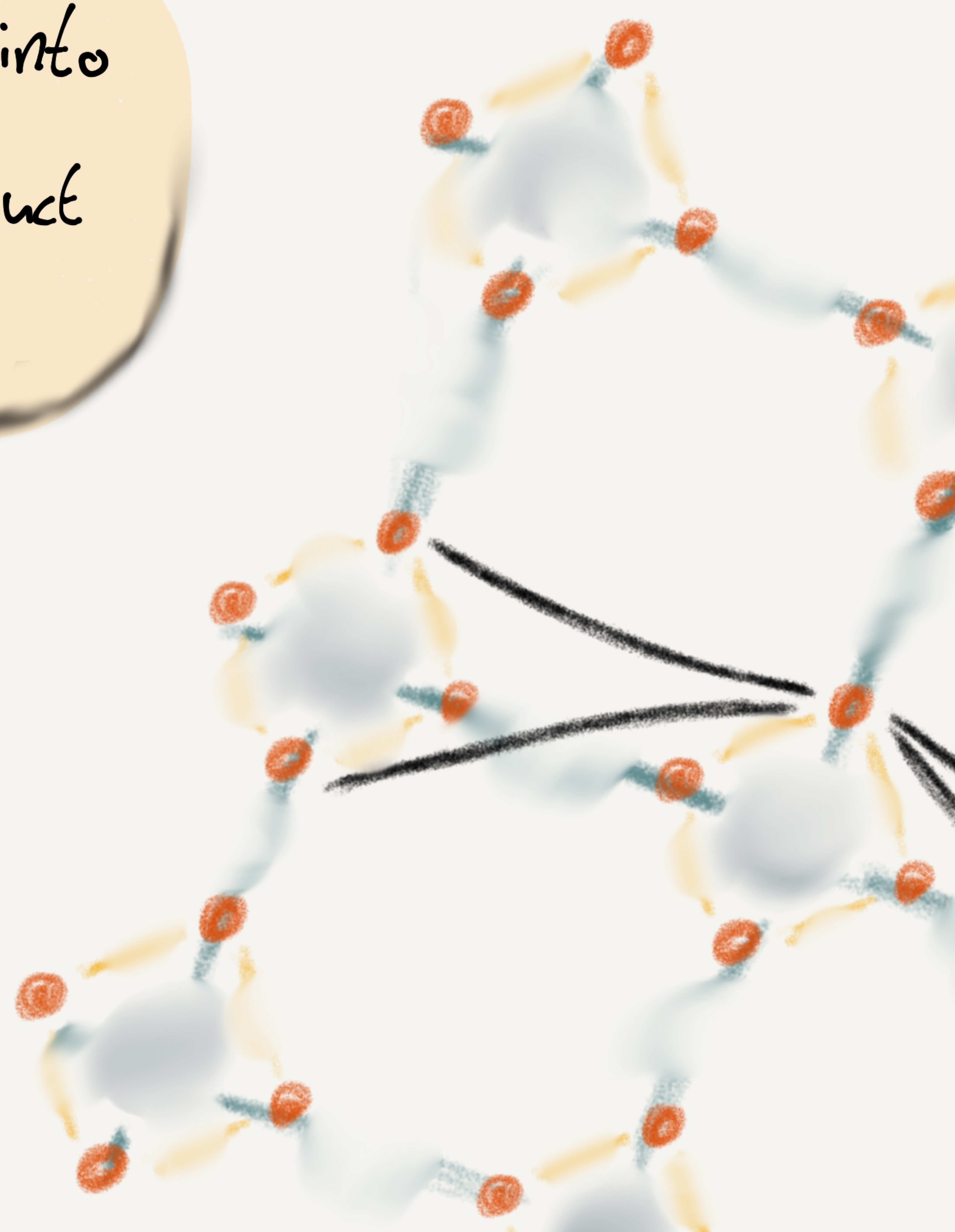
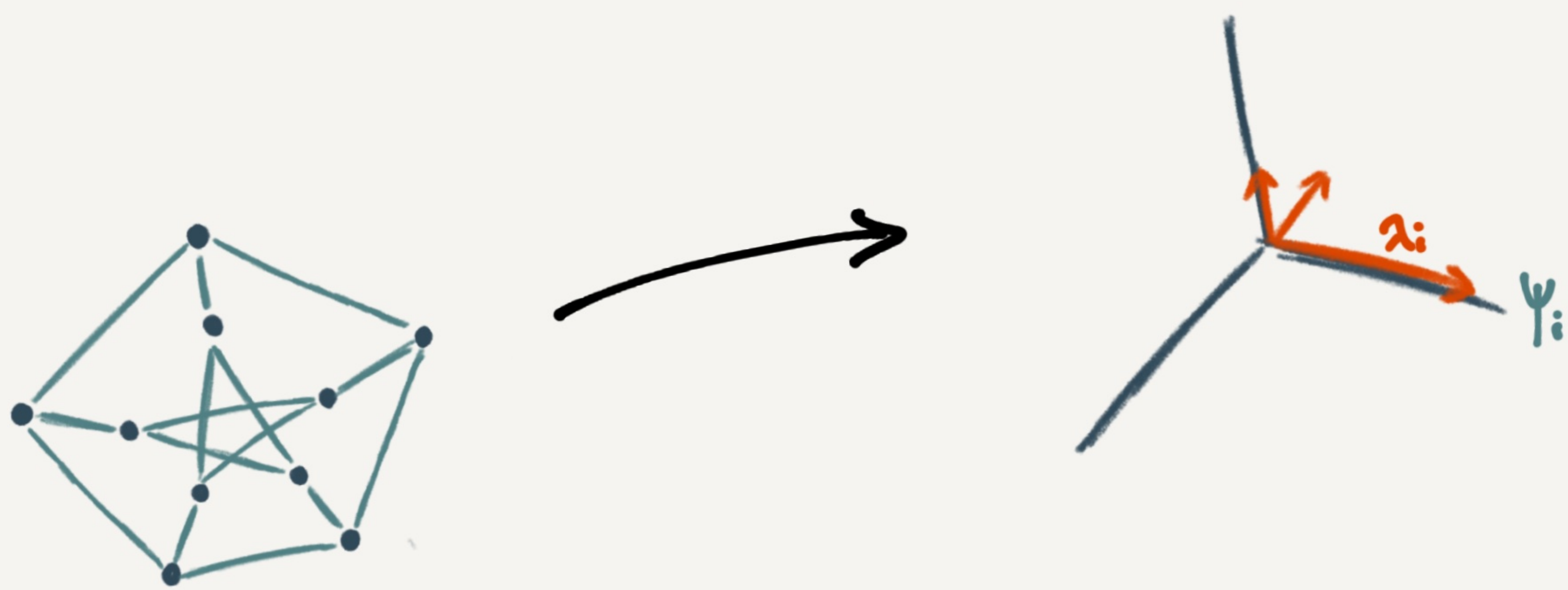
Tel Aviv University

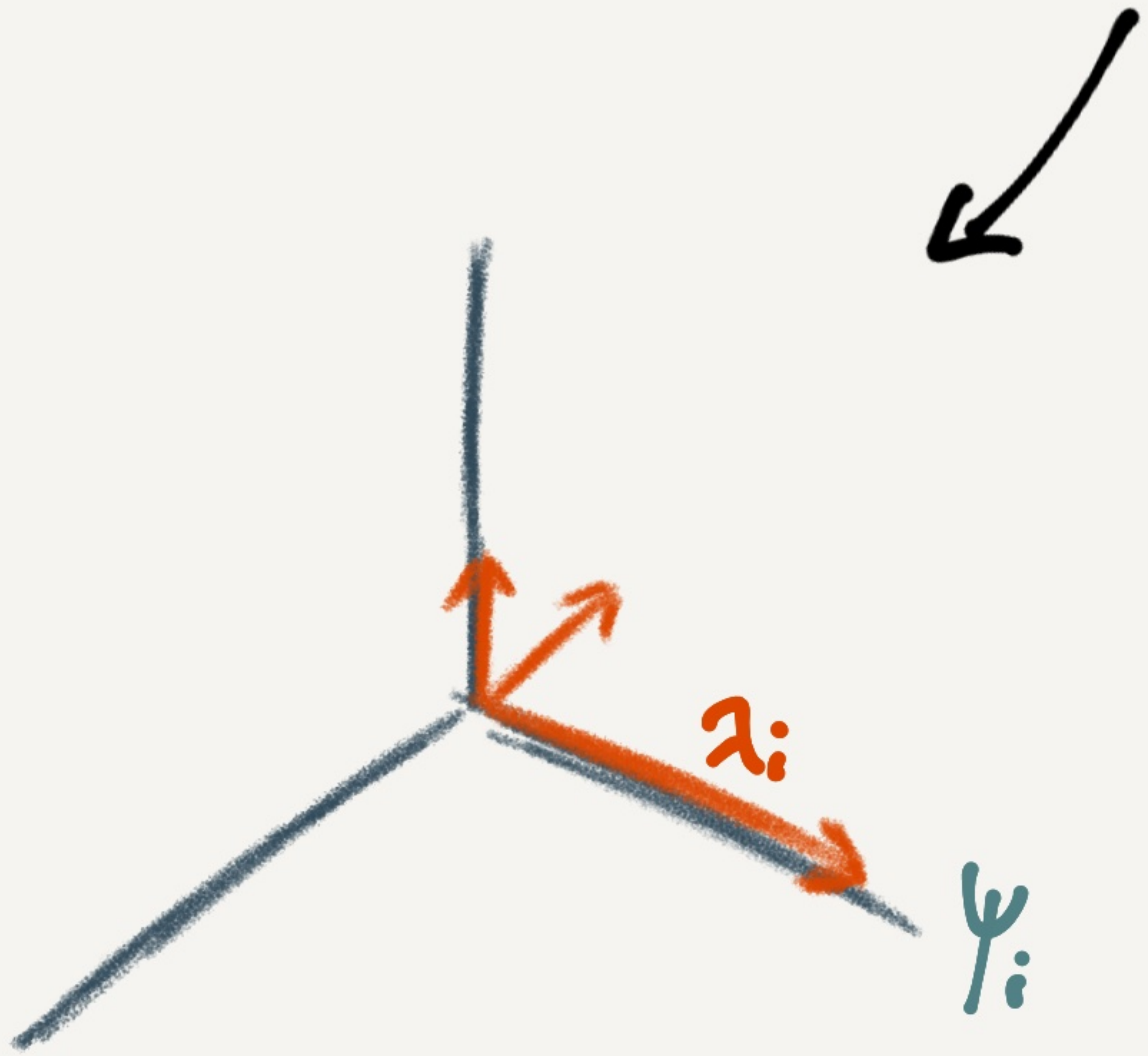


Spectral Expanders Graphs 101



G

A_G

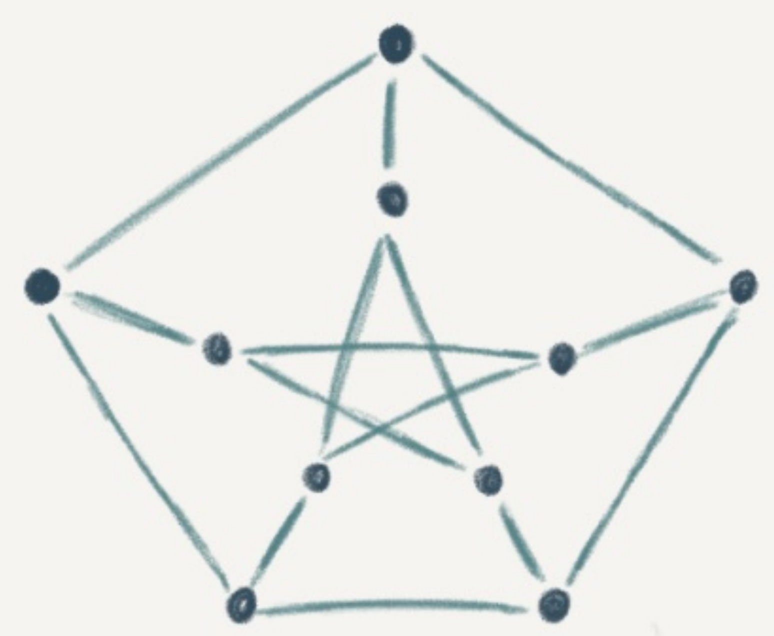


eigenvalue

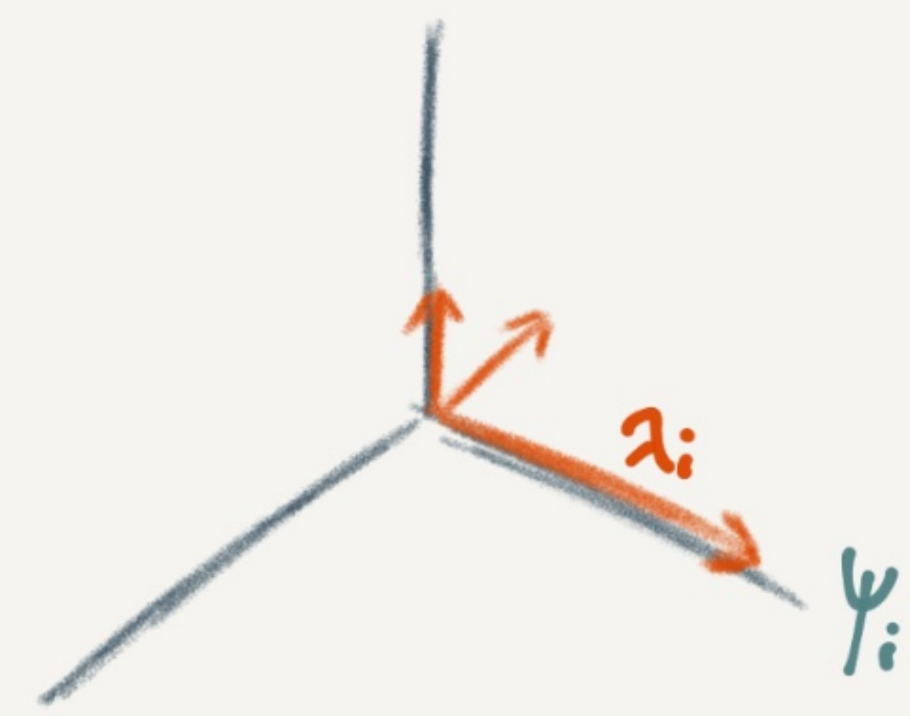
$$A_G = \sum \lambda_i \psi_i \psi_i^T$$

eigenvector

$$\text{Spec}\, G = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$$



G



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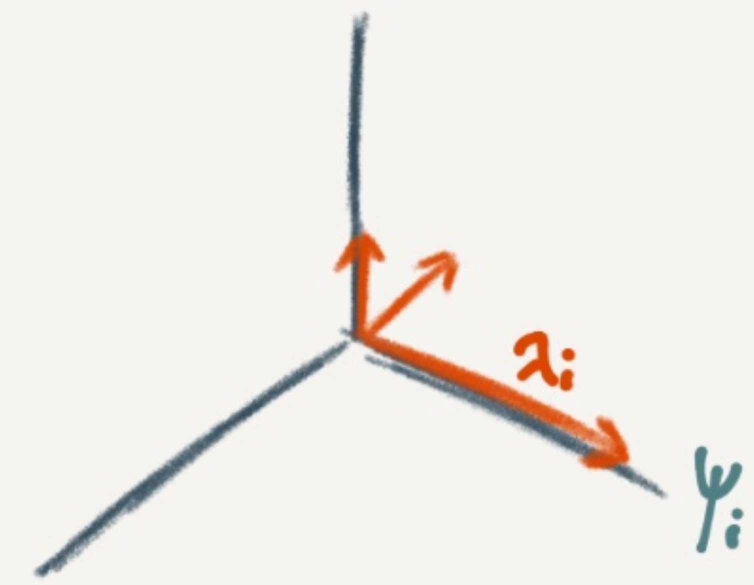
$|\lambda_n| \leq \lambda_1$

G d-regular $\iff$ $\lambda_1 = d$

G



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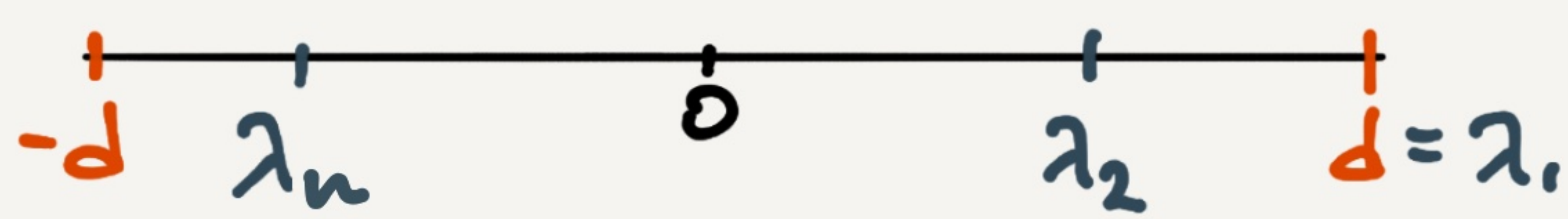
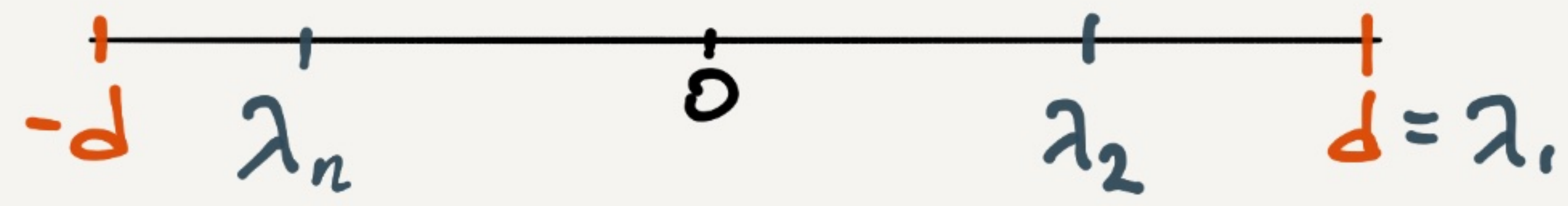


Def. $\lambda_G \triangleq \max\{\lambda_2, |\lambda_n|\}$ is called the spectral expansion of G.

Def. $\lambda_G \triangleq \max\{\lambda_2, |\lambda_n|\}$ is called the **spectral expansion** of G.



λ_G small $\Longrightarrow$ great things

Edge expansion

Expander Chernoff bound

Random walks mix quickly

Expander mixing lemma

Thm [AB '91]. $\forall \varepsilon > 0$ only finitely many d-regular graphs G satisfy

$$\lambda_G < 2\sqrt{d-1} - \varepsilon$$

Def. A d-regular graph G is **Ramanujan** if

$$\lambda_G \leq 2\sqrt{d-1}.$$

Thm [F '08, ...]. A typical d-regular graph is nearly Ramanujan $\left(2\sqrt{d-1} + o(1)\right)$.

Constructing & analyzing expanders

Number/group theory

[LPS'88, Mar'88]

Ramanujan !

Combinatorics & linear algebra

ZigZag [RVW'02...]

Lifting [BL'06...]

⋮

Analytic machinery (~ free probability)

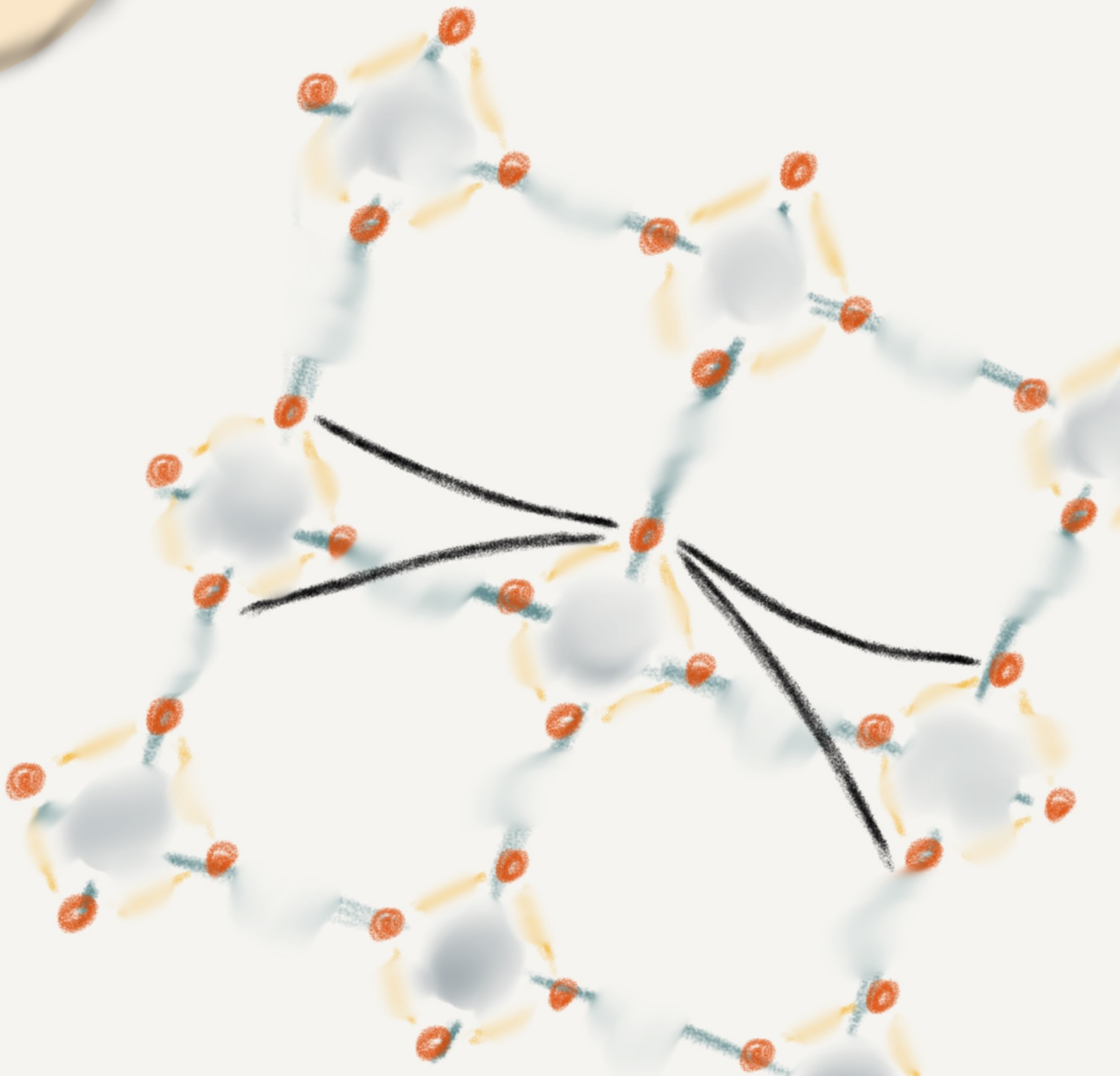
$\forall d$ [MSS'15...]

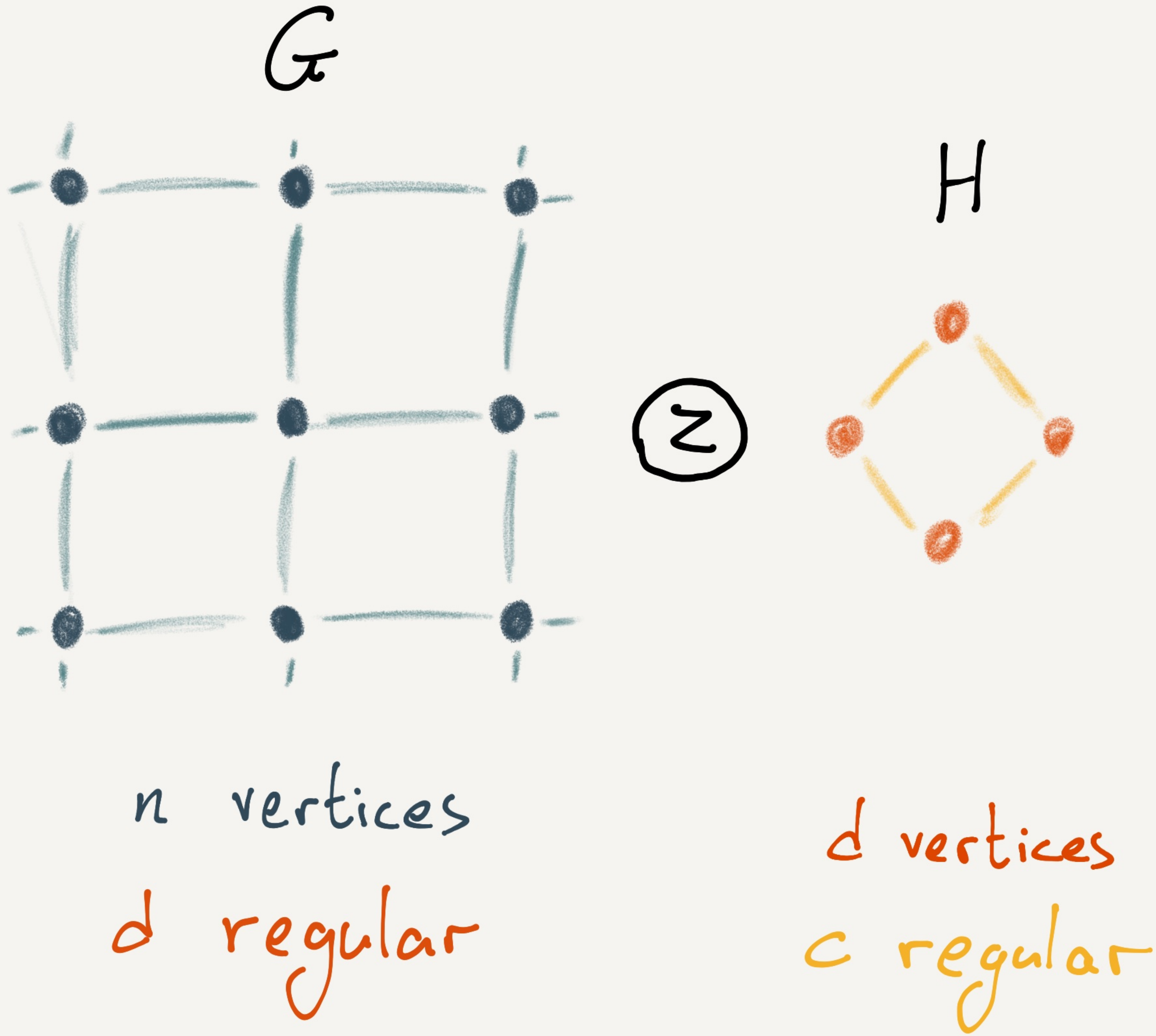
One-sided / Bipartite Ramanujan:

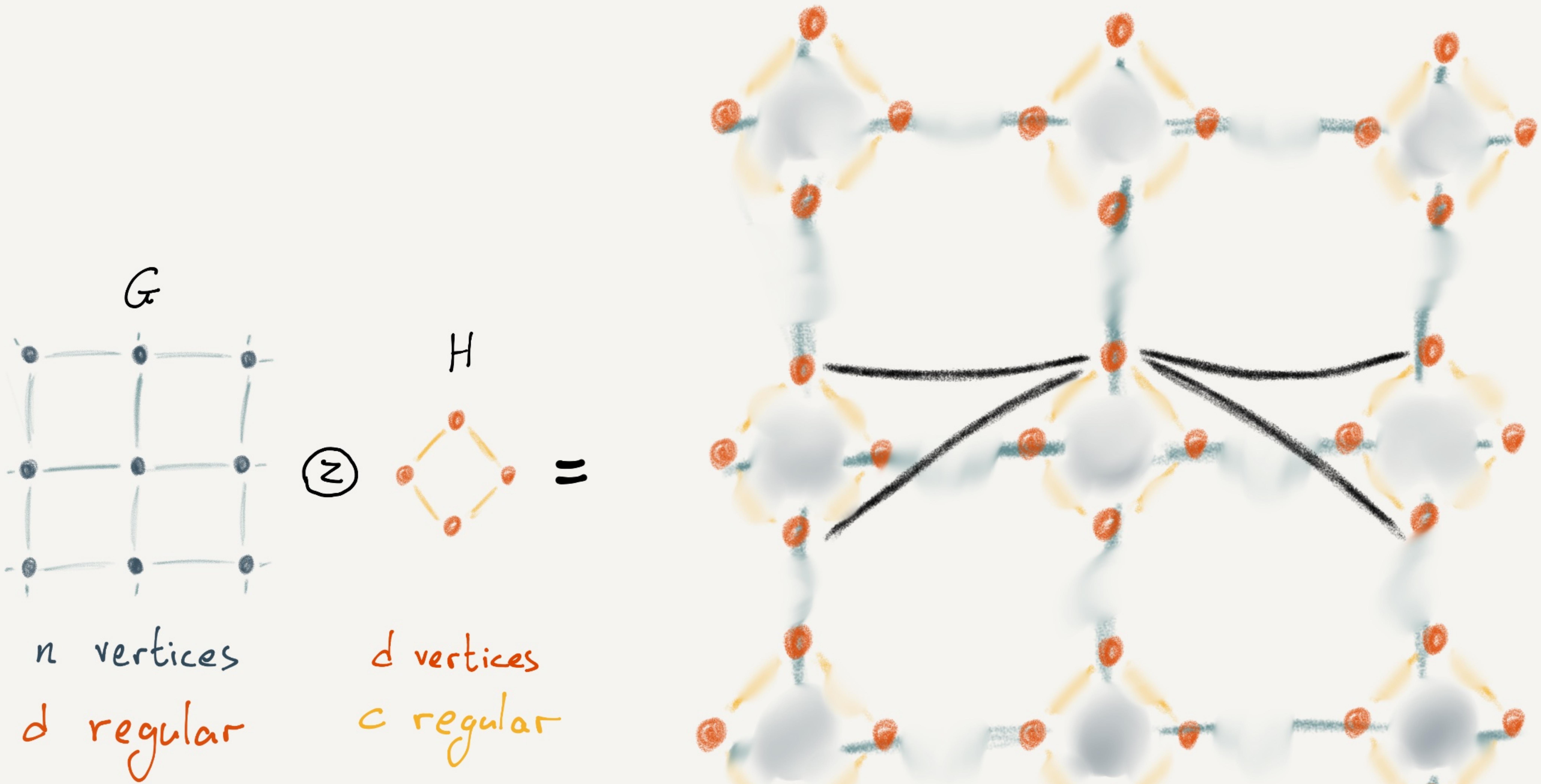
$\lambda_2 \leq 2\sqrt{d-1}$

The Zig-Zag Product

[Reingold-Vadhan-Wigderson '02]

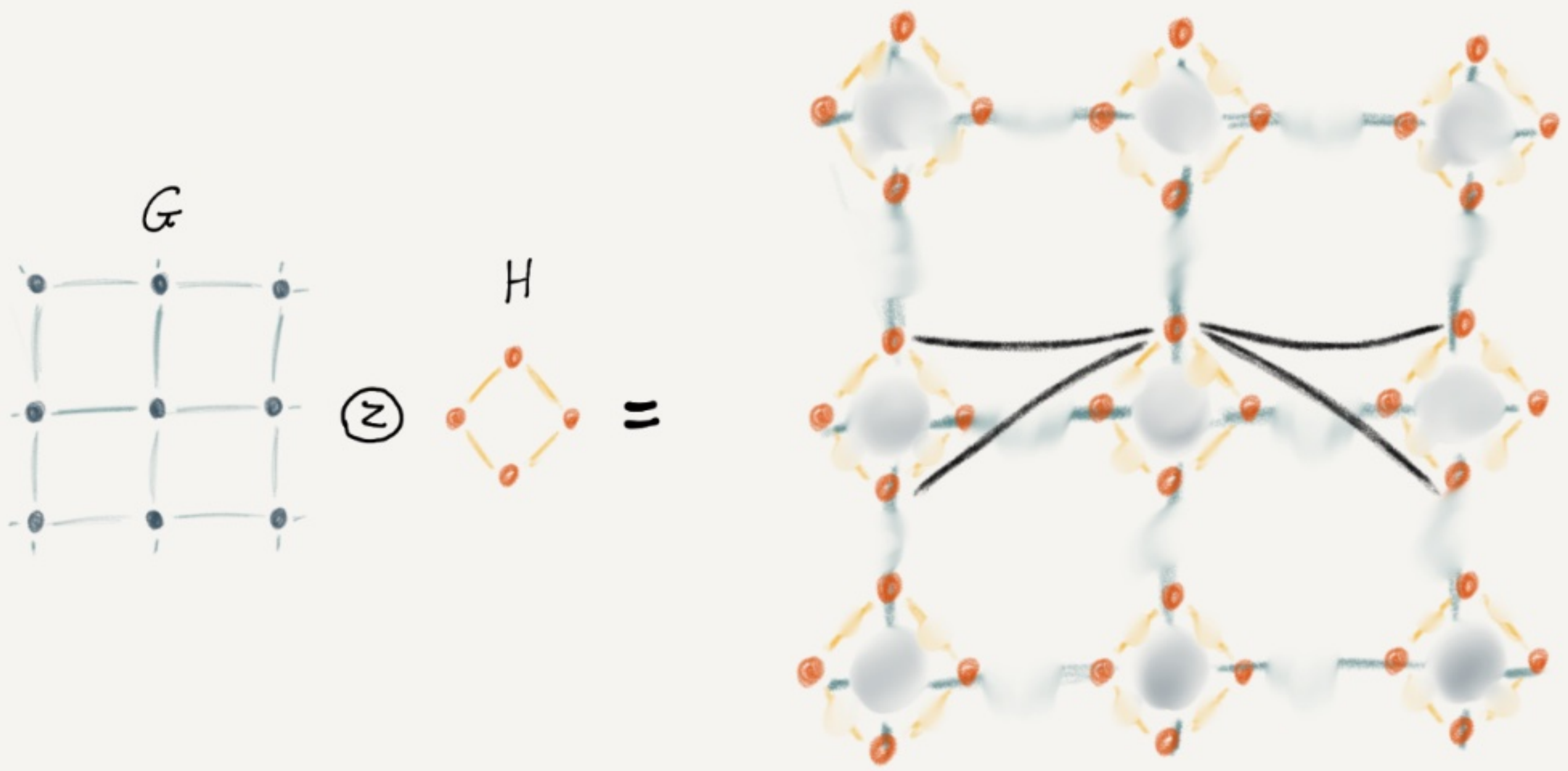



G
H
z
n vertices
d regular
d vertices
c regular



$n \cdot d$ vertices

c^2 regular

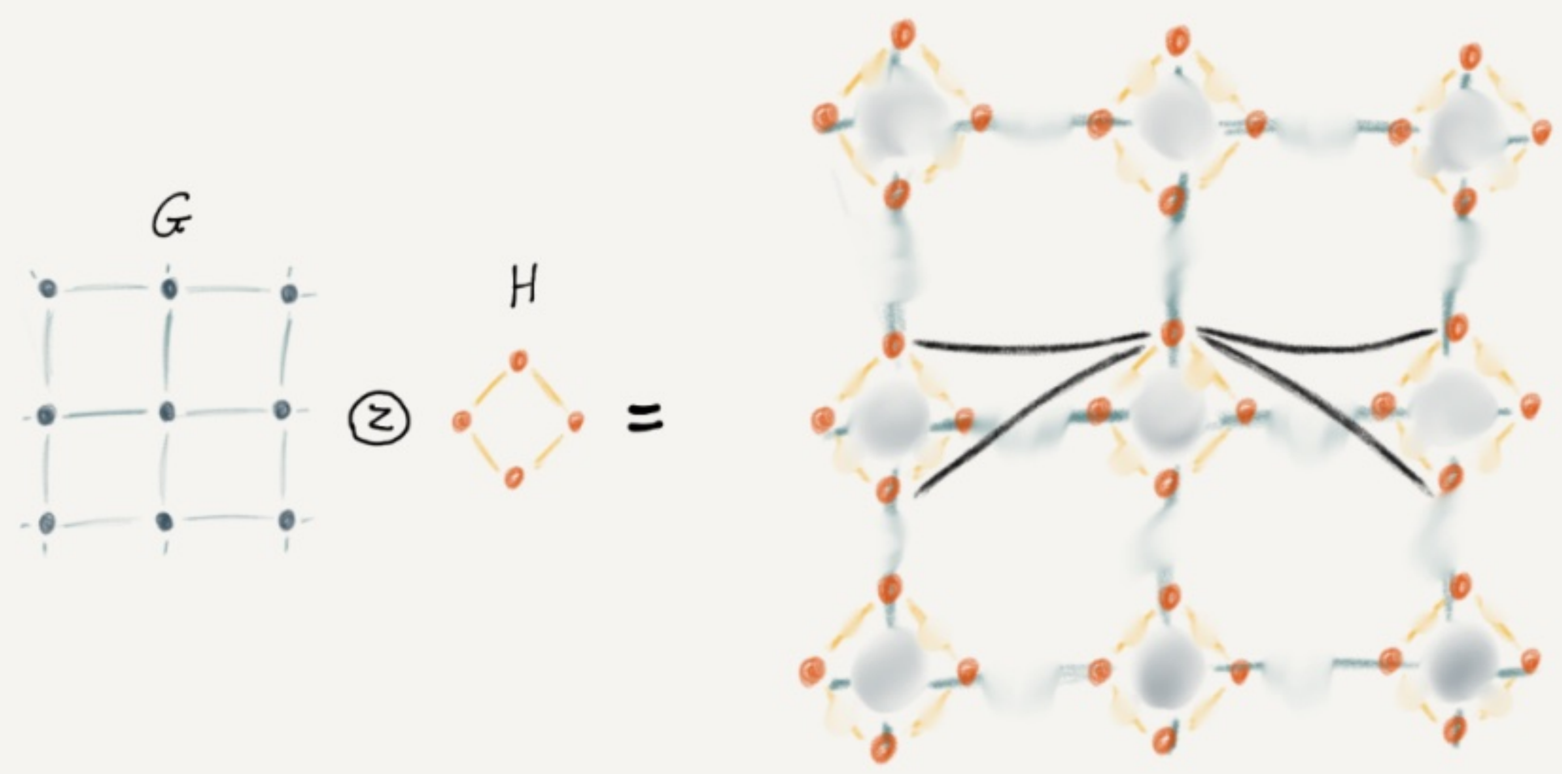


$\frac{\lambda}{deg} \in [0,1]$

	$\lvert V \rvert$	degree	ω
G	n	d	ω_G
H	d	c	ω_H
$G ⓩ H$	$n \cdot d$	c^2	?

<u>Thm</u> [RVW].

$$\forall\, G \;\&\; H \qquad \omega_{G ⓩ H} \lesssim \omega_G + \omega_H$$



	$\lvert V \rvert$	degree	ω
G	n	d	ω_G
H	d	c	ω_H
$Z = G ⓩ H$	$n \cdot d$	c^2	?

Thm [RVW]. $\omega_{G ⓩ H} \lesssim \omega_G + \omega_H$

Question.

Fix H. How does $\omega_{G ⓩ H}$ "really" behave?

worst case

Best case

typical

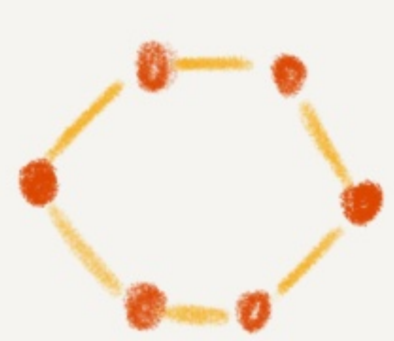
Question.

Fix H. How does $\omega_{G \textcircled{z} H}$ "really" behave?

worst case

Best case

typical

H	[AB]	[RVW]	[Reality]
	$\frac{\sqrt{3}}{2} \sim 0.86$	1	$0.96\ldots$
	$\frac{2\sqrt{8}}{9} \sim 0.63$	$\frac{1+\sqrt{17}}{6} \sim 0.85$	$0.79\ldots$

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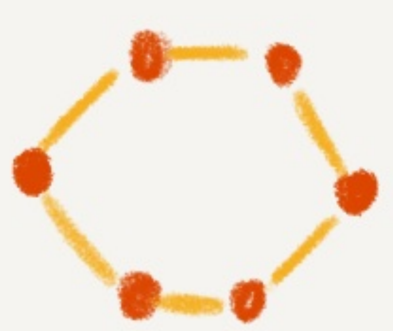
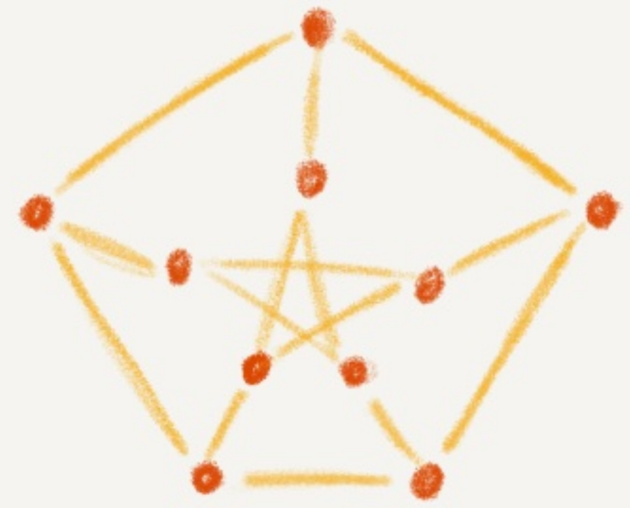
H	[AB]	[RVW]	[Reality]	
	$\frac{\sqrt{3}}{2} \sim 0.86$	1	$0.96\ldots$	$\frac{1+\sqrt{3}}{\sqrt{8}}$
	$\frac{2\sqrt{8}}{9} \sim 0.63$	$\frac{1+\sqrt{17}}{6} \sim 0.85$	$0.79\ldots$	not for the faint of heart

Question.

Fix H. How does $\omega_{G \text{ⓩ} H}$ "really" behave ?

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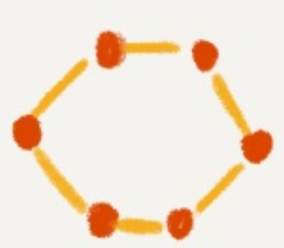
H	[AB]	[RVW]	[Reality]
	$\frac{\sqrt{3}}{2} \sim 0.86$	1	$0.96\ldots$ $\frac{1+\sqrt{3}}{\sqrt{8}}$
	$\frac{2\sqrt{8}}{9} \sim 0.63$	$\frac{1+\sqrt{17}}{6} \sim 0.85$	$0.79\ldots$

$$\frac{1}{3\sqrt{\frac{11}{33+\sqrt[3]{11(275-4\sqrt{11})}+\sqrt[3]{11(275+4\sqrt{11})}}}}$$

Question. Fix H. How does $\omega_{G \circledcirc H}$ "really" behave?

worst case

Best case

typical

Main Thm.

$\forall^*$ vertex-transitive c-regular H and

$\forall$ d-regular G

$$\omega_{G \circledcirc H} \geq \min_{x > c^2} Z_H(x) - o(1)$$

where

characteristic polynomial of H^2

$$Z_H(x) = \frac{x}{c^2} \cdot \sqrt{1 - \frac{d \cdot h(x)}{x \cdot h'(x)}}$$

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$$\omega_{G \circledcirc H} \geq \min_{x > c^2} Z_H(x) - o(1)$$

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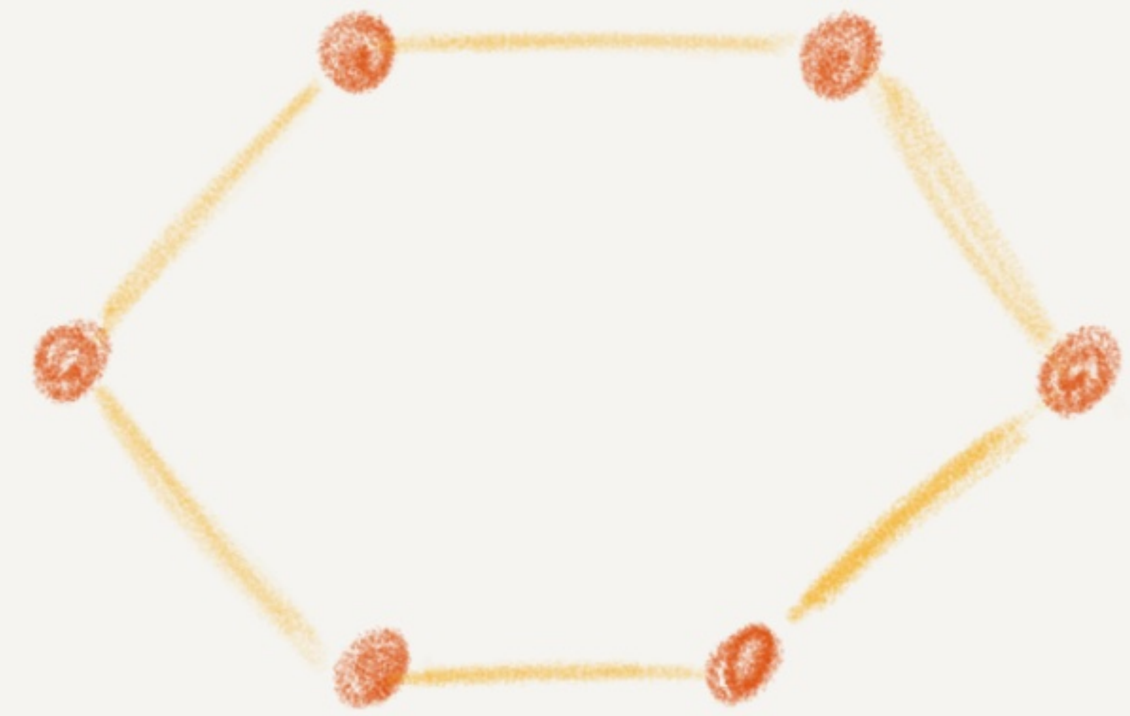
$$Z_H(x) = \frac{x}{c^2} \cdot \sqrt{1 - \frac{d \cdot h(x)}{x \cdot h'(x)}}$$

Moreover, the bound is "one-sided" tight:

$\forall H, n \ \exists G$ of size n s.t.

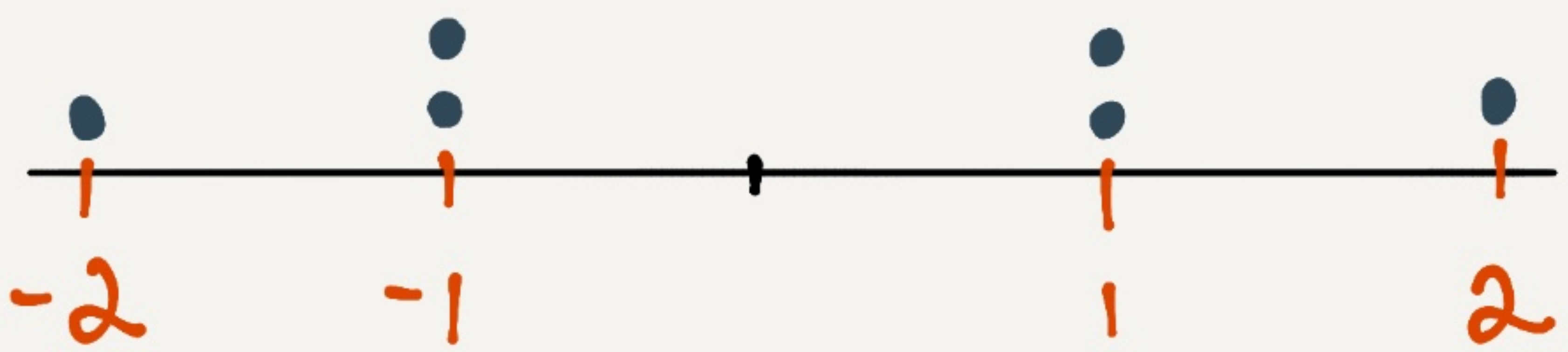
$$(\omega_2)_{G \circledcirc H} \leq \min_{x > c^2} Z_H(x)$$

Example

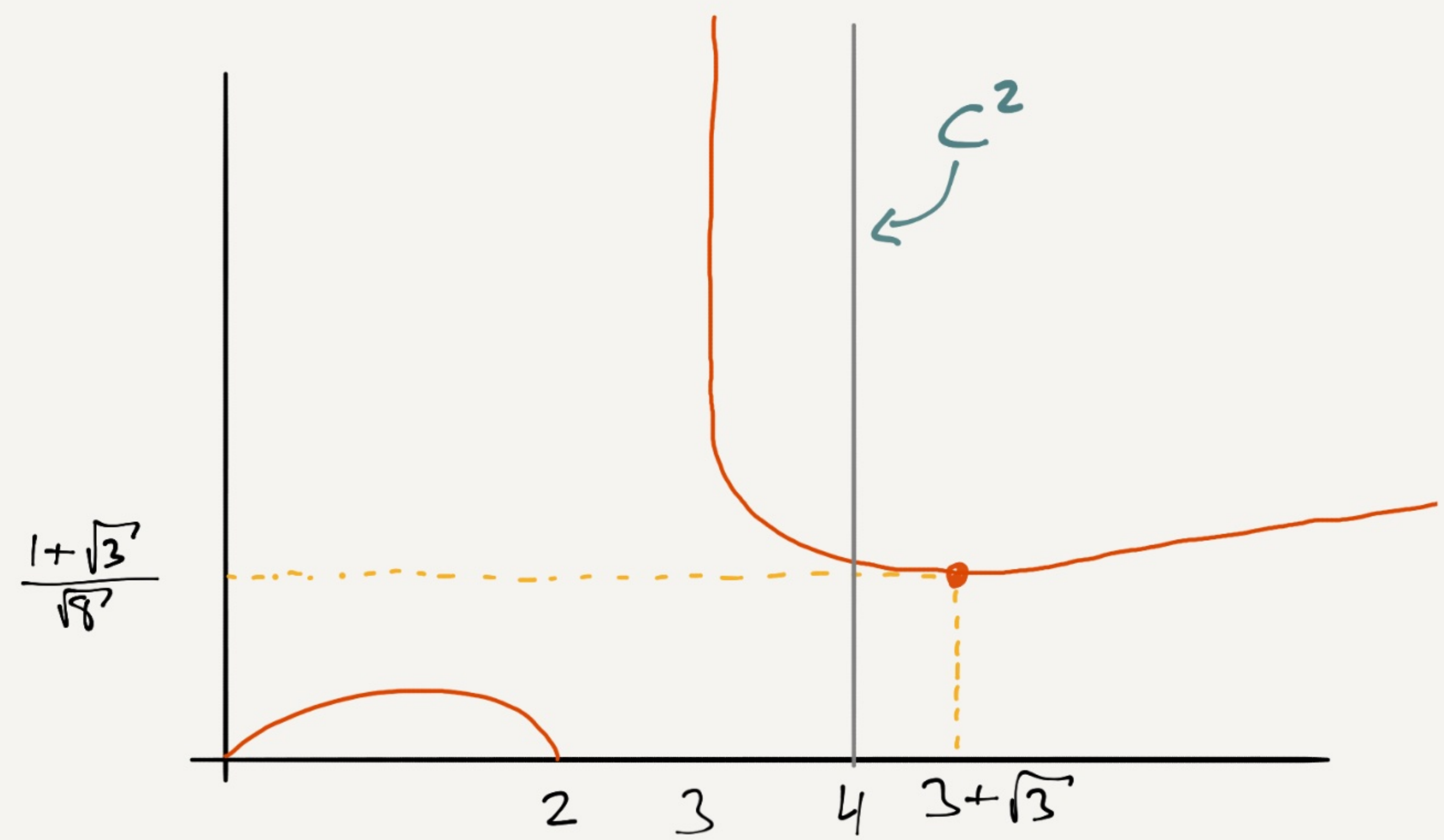


characteristic polynomial of H^2

$$\left[\, Z_H(x) = \frac{x}{c^2}\cdot\sqrt{1-\frac{d\cdot h(x)}{x\cdot h'(x)}} \,\right]$$



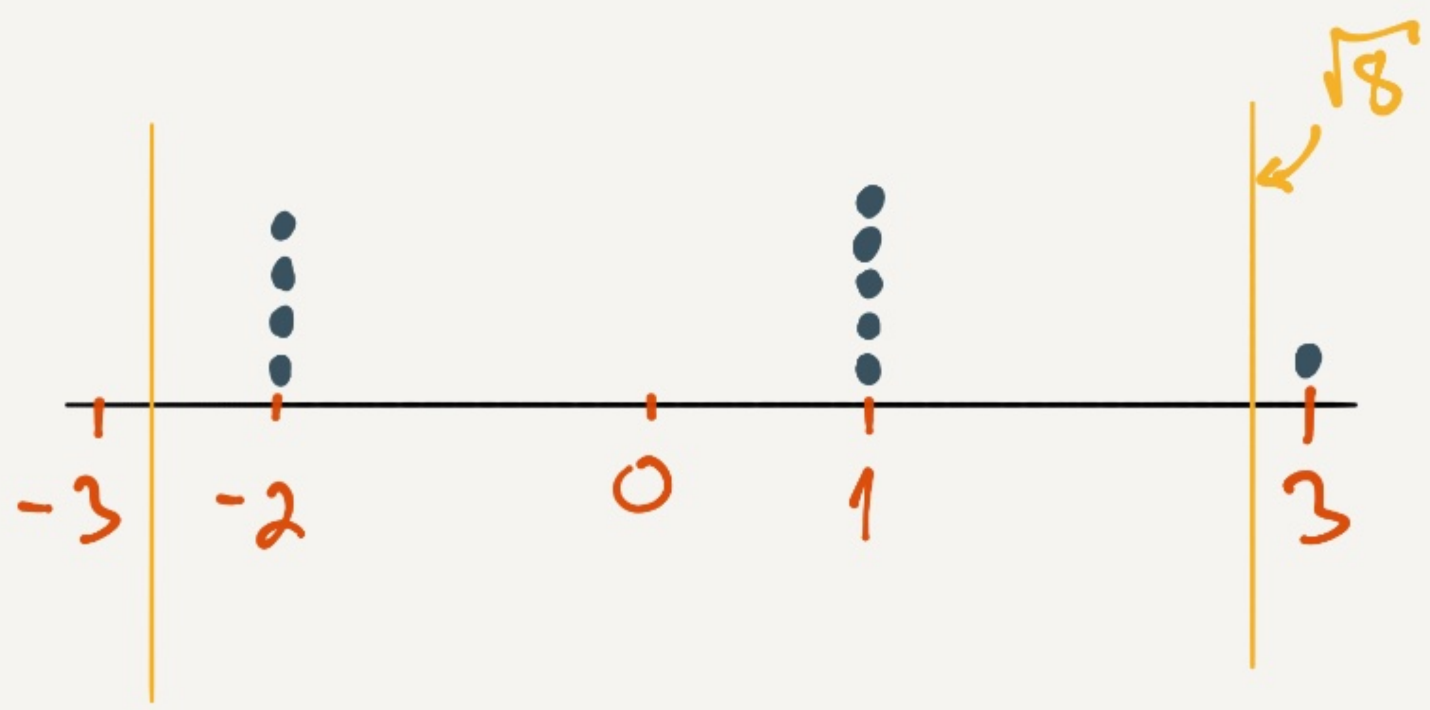
$$h(x) = (x-4)^2(x-1)^4 \quad\Longrightarrow\quad Z_{C_6}(x) = \sqrt{\frac{x(x-2)}{8(x-3)}}$$



Example

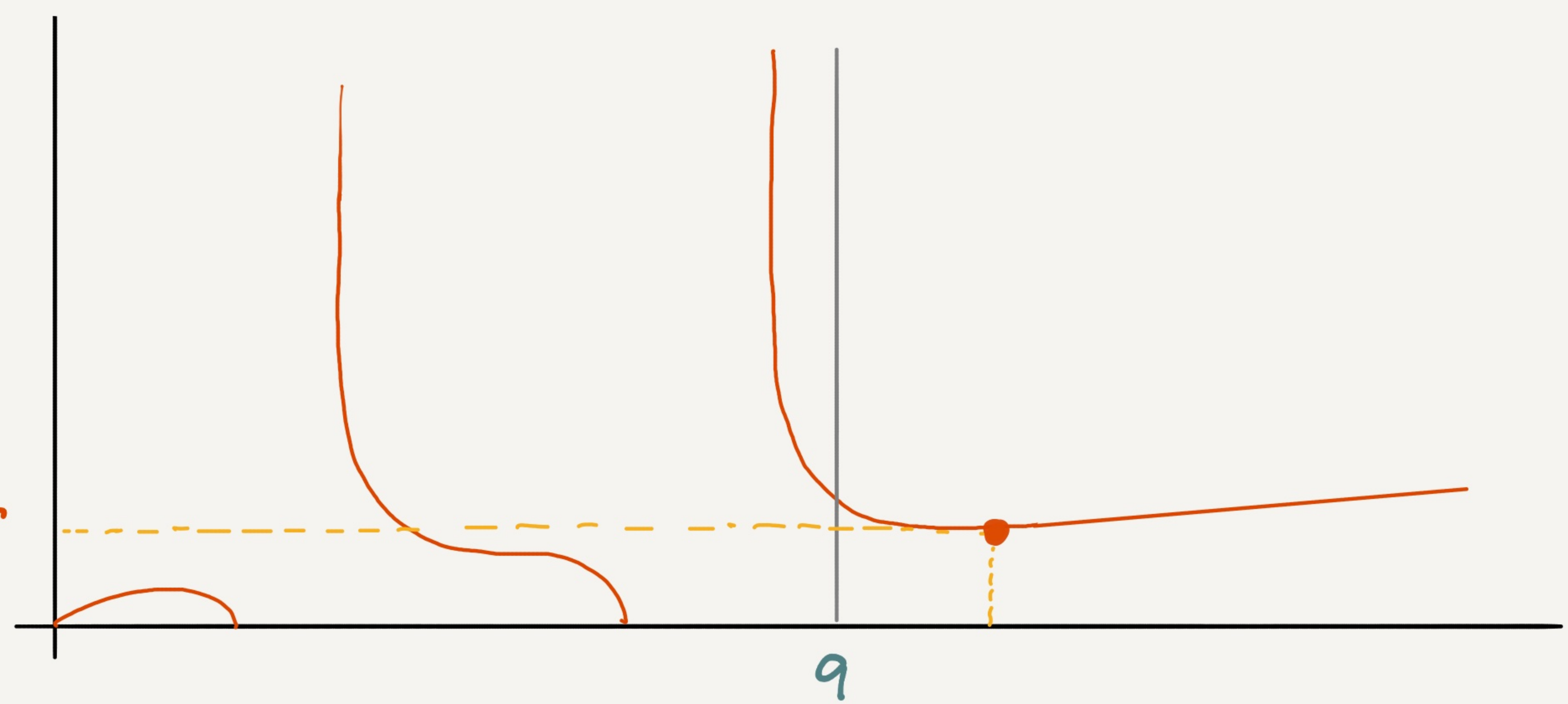
$$\left[\; Z_H(x) = \frac{x}{c^2} \cdot \sqrt{1 - \frac{d \cdot h(x)}{x \cdot h'(x)}} \;\right]$$

characteristic polynomial of H^2



$$\Longrightarrow \quad Z_{Pet}(x) = \sqrt{\frac{x(x^2-9x+12)}{27(x^2-11x+22)}}$$

$$\frac{1}{3\sqrt{\dfrac{11}{33+\sqrt[3]{11(275-4\sqrt{11})}+\sqrt[3]{11(275+4\sqrt{11})}}}}$$



Main Thm.

$\forall^*$ vertex-transitive c-regular H and

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$$\omega_{G \circledS H} \geq \min_{x > c^2} Z_H(x) - o(1)$$

where

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Analytic combinatorics

The bound is "one-sided" tight:

$\forall H, n \; \exists \, G$ of size n s.t.

$$(\omega_2)_{G \circledS H} \leq \min_{x > c^2} Z_H(x)$$

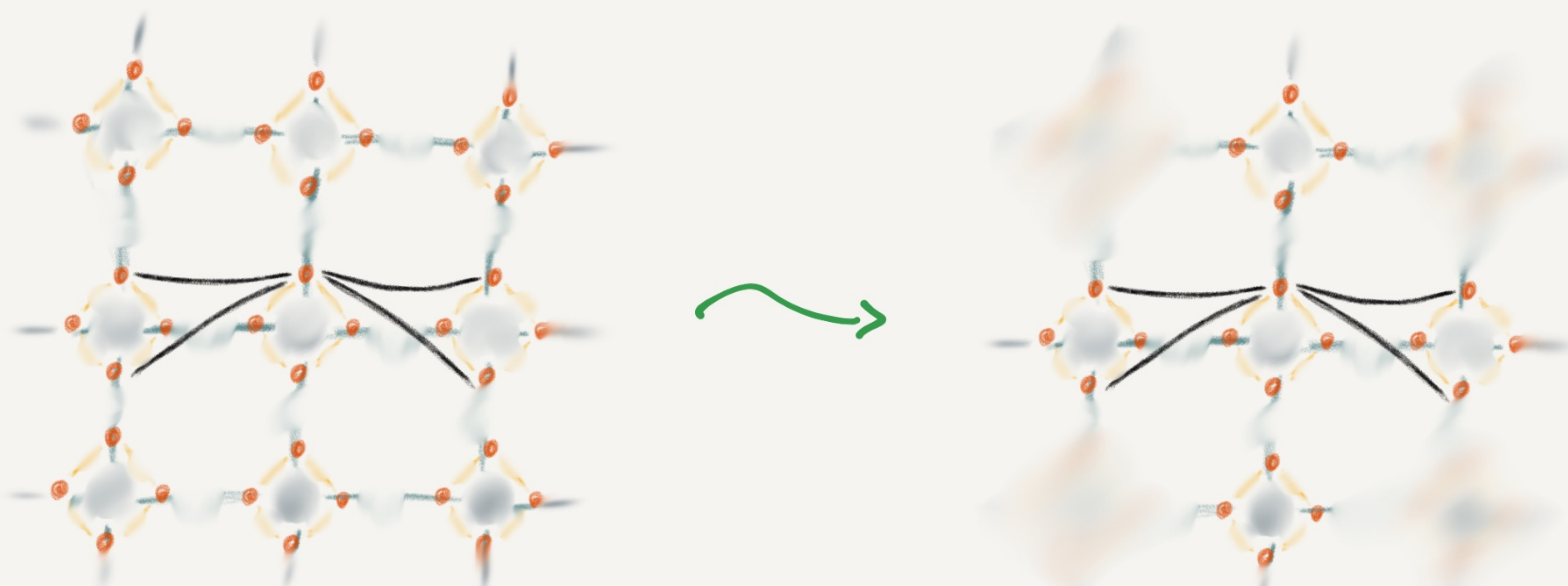
(Finite) Free Probability

Proof outline.

By the trace method

$$\lambda(G \circledcirc H) \geqslant (\text{length-}k \text{ cycles in } G \circledcirc H)^{1/k} - o(1)$$

$$\geqslant (\text{length-}k \text{ cycles in } T \circledcirc H)^{1/k} - o(1)$$



Counting cycles in $T \circledz H$.

$$C(z) = \frac{1}{1 - S(z)}$$

Let $S_{T\circledz H}(z)$ be the generating function of simple cycles originate at the root:

$$S_{T\circledz H}(z) = \sum_k S_k(T \circledz H)\, z^k$$

Using the symbolic method we prove

$$S_{T\circledz H}(z) = z^2 \cdot A_H\left(S_{T\circledz H}(z)\right)$$

$$S_{T\circledast H}(z) = z^2 \cdot A_H\left(S_{T\circledast H}(z)\right)$$

Let $\tilde{H}_v = H^2 \setminus \{$  $\}$

length-2 paths not through v

$$A_H(z) = \sum z^k \cdot \#\left\{\begin{array}{l}\text{length-}k\text{ paths from}\\ \Gamma(v) \text{ to } \Gamma(v) \text{ in } \tilde{H}_v\end{array}\right\}$$

Combinatorial & linear algebraic considerations

$$\vdots$$

$$= \frac{1}{z} - \frac{1}{G_{H^2}\left(\frac{1}{z}\right)}$$

$$\frac{h'\left(\frac{1}{z}\right)}{d \cdot h\left(\frac{1}{z}\right)}$$

$$S_{T\boxtimes H}(z) = z^2 \cdot A_H\left(S_{T\boxtimes H}(z)\right)$$

$$A_H(z) = \frac{1}{z} - \frac{1}{G_{H^2}\left(\frac{1}{z}\right)}$$

<u>Theorem</u>. Assume $\zeta(z)$ satisfies

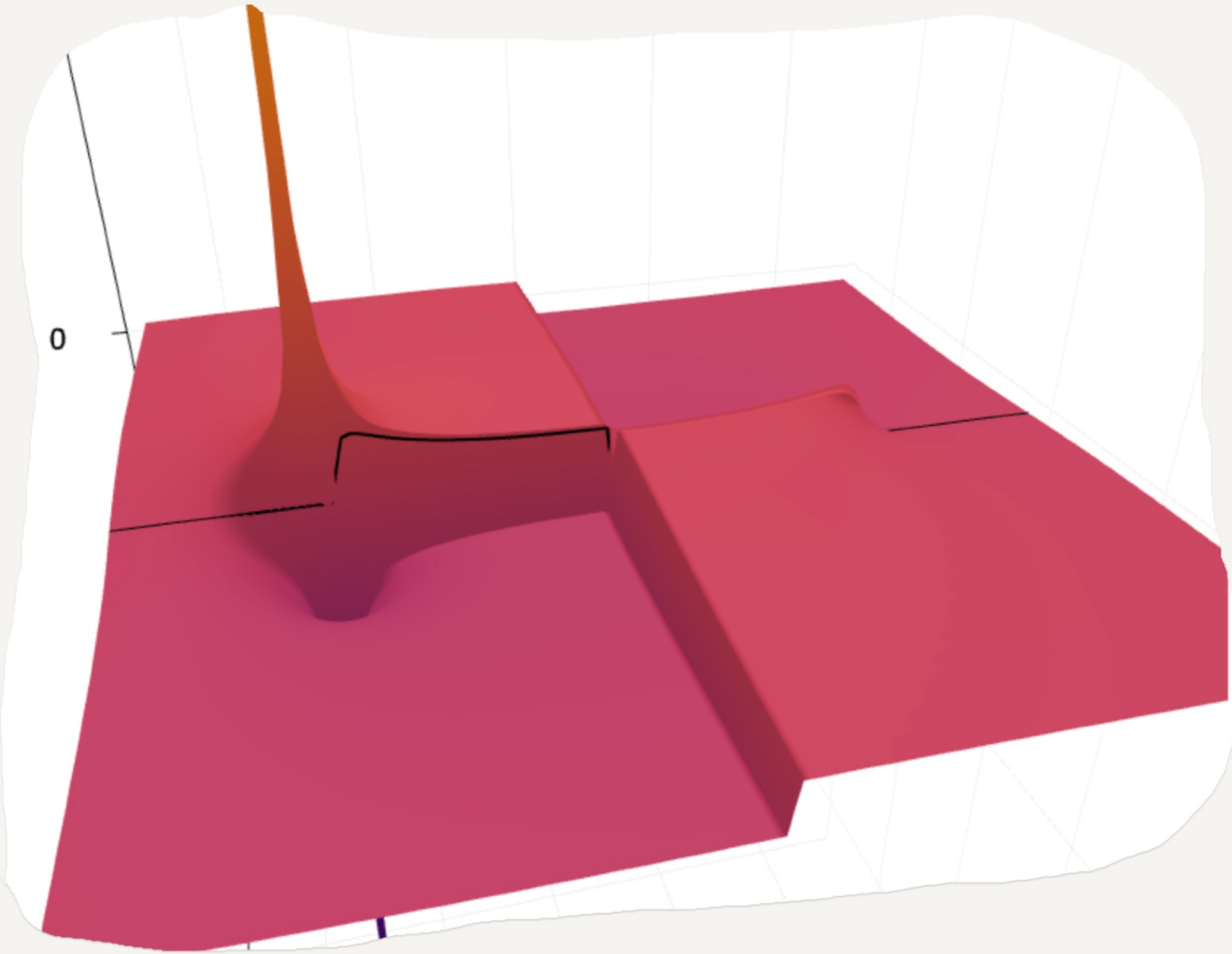
$$\zeta(z) = z \cdot \phi(\zeta(z))$$

Then.(*)

$$[z^n]\,\zeta(z) \approx \phi'(\tau)^n$$

where τ is the solution to the characteristic equation

$$\phi(u) = u \cdot \phi'(u)$$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The big picture

Structure

Number / group theory

we are just good at it

Combinatorics & linear algebra

Spectrum

Free Probability & Analytic Combinatorics

What's next?

Applications: Seems to be great for compositions

* Better INW?
* Relaxed LCCs? [KM'24, CY'24]

⋮

[MSS'15] Ramanujan graphs
[CM'23] Rotating expanders
[CCMP'22] Derandomized squaring
[CCM'24] Zig Zag

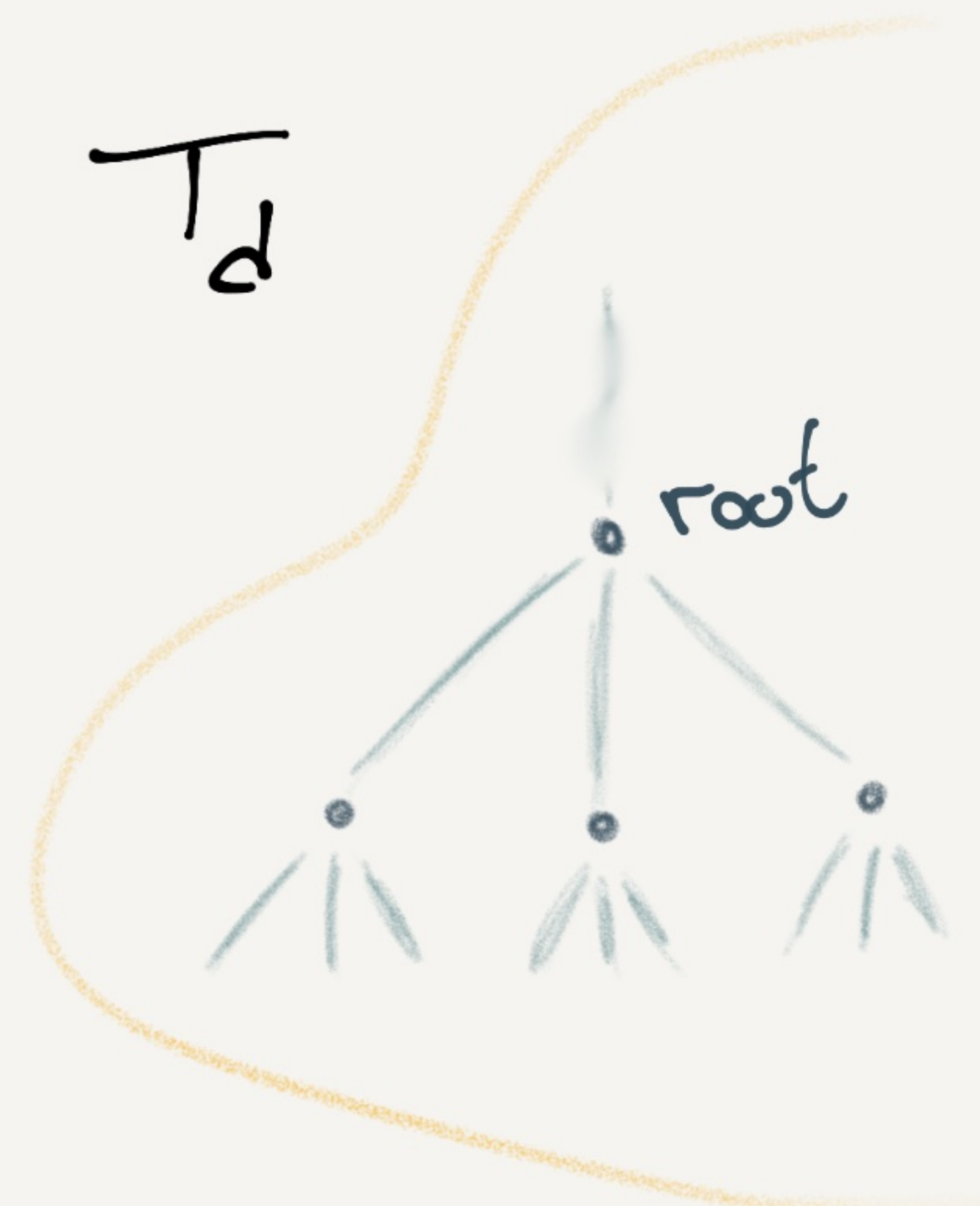
Fundamental questions:

* ∞ vs. finite
* The "free method"

⋮

Bonus: Analytic explanation of the magical $2\sqrt{d-1}$

$\mathcal{C}$: class of cycles in the truncated T_d
originate at the root



$$\mathcal{C} = SEQ\left(\{1,2,\dots,d-1\} \times \mathcal{C} \times \uparrow\right)$$

$$\Downarrow$$

$$C(z) = \frac{1}{1-(d-1)z^2 C(z)} \implies \begin{pmatrix} D(z^2) = C(z) \\ E(z) = z\,D(z) \end{pmatrix}$$

$$E(z) = z \cdot \phi\big(E(z)\big)$$

$$\phi(u) = \frac{1}{1-(d-1)z}$$

$$E(z) = z \cdot \phi(E(z)) \qquad \phi(u) = \frac{1}{1-(d-1)z}$$

Characteristic equation $\left(\phi(u) = u\phi'(u)\right)$:

$$\underbrace{\frac{1}{1-(d-1)u}}_{\phi(u)} = u \cdot \underbrace{\frac{(d-1)}{(1-(d-1)u)^2}}_{\phi'(u)}$$

Solution: $\tau = \frac{1}{2(d-1)} \implies \phi'(\tau) = 4(d-1)$

$$\implies [z^n]E(z) \approx (4(d-1))^n \implies [z^n]C(z) \approx \left(2\sqrt{d-1}\right)^n.$$