Algebraic-Geometric Codes **Fall 2024/5**

Problem Set 0: Ring Theory Refresher

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Remark. All the rings in these exercises are commutative with 1. A *ring homomorphism* $f\colon R \to S$ is a map that satisfies

(a) $f(1) = 1$.

(b) $f(a+b) = f(a) + f(b)$ for all $a, b \in R$.

(c) $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Problem 1. *Let $I, J \trianglelefteq R$. Determine if the following sets are ideals of R. If not, find the smallest ideal which contains it.*

(a) $I \cap J$.

(b) $I \cup J$.

(c) $I + J := \{a + b \mid a \in I,\, b \in J\}$.

(d) $I * J := \{ab \mid a \in I,\, b \in J\}$.

If I, J are prime, which of the above is a prime ideal?
If I, J are maximal, which of the above is a maximal ideal?
If I, J are principal, which of the above is a principal ideal?

Problem 2. *Let R, S be rings and let $f\colon R \to S$ be a ring homomorphism. Which of the following statements are true? No proof needed.*

(a) If $I \triangleleft R$ then $f(I) \triangleleft S$.

(b) If $J \triangleleft S$ then $f^{-1}(J) \triangleleft R$.

(c) If $I \in \mathrm{Spec}(R)$ then $f(I) \in \mathrm{Spec}(S)$.

(d) If $J \in \mathrm{Spec}(S)$ then $f^{-1}(J) \in \mathrm{Spec}(R)$.

(e) If $I \in \mathrm{MaxSpec}(R)$ then $f(I) \in \mathrm{MaxSpec}(S)$.

(f) If $J \in \mathrm{MaxSpec}(S)$ then $f^{-1}(J) \in \mathrm{MaxSpec}(R)$.

(g) If I is a principal of R then $f(I)$ is a principal ideal of S.

(h) If J is a principal of S then $f^{-1}(J)$ is a principal ideal of R.

Problem 3. *Characterize the ideals in R/I, i.e. describe the ideals in R/I in terms of the ideals of R.*

Problem 4. *Let k be a field and let $a := (a_1, \dots, a_n) \in k^n$. In the recitation we claimed that the substitution homomorphism*

$$\varphi_a \colon k[X_1, \dots, X_n] \longrightarrow k$$
$$f(X_1, \dots, X_n) \longmapsto f(a_1, \dots, a_n)$$

is a surjective ring homomorphism with kernel

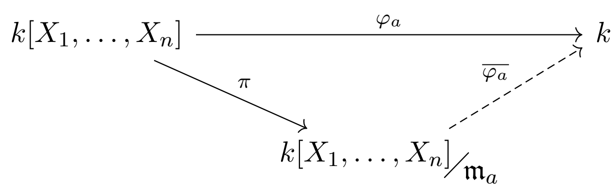
$$\ker(\varphi_a) = \langle X_1 - a_1, \dots, X_n - a_n \rangle =: \mathfrak{m}_a.$$

It is clear that $\ker(\varphi_a) \supseteq \mathfrak{m}_a$. *For the opposite inclusion, it is sufficient to prove that* $\mathfrak{m}_a \in \operatorname{MaxSpec}(k[X_1, \dots, X_n])$. *Show it by the following steps.*

(a) *Show that there exists a unique ring homomorphism*

$$\overline{\varphi_a} \colon k[X_1, \dots, X_n]/\mathfrak{m}_a \to k$$

such that $\varphi_a = \overline{\varphi_a} \circ \pi$, *i.e. the following diagram commutes:*



(b) *Show that* $\overline{\varphi_a}$ *is surjective.*

(c) *Show that* $\overline{\varphi_a}$ *is injective. Hint: first show that* $\pi(g(X_1, \dots, X_n)) = \pi(g(a_1, \dots, a_n))$ *for every* $g \in k[X_1, \dots, X_n]$.

(d) *Conclude that* $\mathfrak{m}_a$ *is a maximal ideal of* $k[X_1, \dots, X_n]$.

Problem 5. *Let $R \subseteq S$ be integral domains. Prove or disprove:*

(a) *Every irreducible element in $R[x]$ is irreducible in $S[x]$.*

(b) *Every irreducible element in $S[x]$ is irreducible in $R[x]$.*

Problem 6. *Let R be an integral domain and let $0 \neq a \in R$.*

(a) *Prove that a is prime $\iff$ $\langle a \rangle$ is a prime ideal of R.*

(b) *Prove that if a is prime then a is irreducible.*

We say that R is a UFD (unique factorization domain) *if every non-zero non-unit element $b \in R$ can be written as $b = p_1 \cdots p_k$ where $p_1, \dots, p_k \in R$ are irreducible ($k \geq 1$), and in addition if $b = q_1 \dots q_m$ where $q_1, \dots, q_m \in R$ are irreducible ($m \geq 1$), then $k = m$ and there exist a permutation $\sigma \colon [k] \to [k]$ and units $u_1, \dots, u_k \in R^\times$ such that $q_i = u_i p_{\sigma(i)}$.*

(c) *Show that if R is a UFD and a is irreducible, then a is prime.*

Problem 7. *Let $S := \mathbb{Z} + x^2\,\mathbb{Z}[x] = \{a + x^2 f(x) \mid a \in \mathbb{Z},\ f(x) \in \mathbb{Z}[x]\}$.*

(a) *Prove that $S \subset \mathbb{Z}[x]$ is a subring.*

(b) *Prove that $x^6 \in S$ can be written as a product of irreducible elements in two different ways. Conclude that S is not a UFD.*

(c) *Find an irreducible element in S which is not prime in S.*