Free Probability & Ramanujan Graphs

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TA: Gal Maor

Some course information.

* homepage: gilcohen.org/2024-fpt
* grade: 50% (~6) problem sets

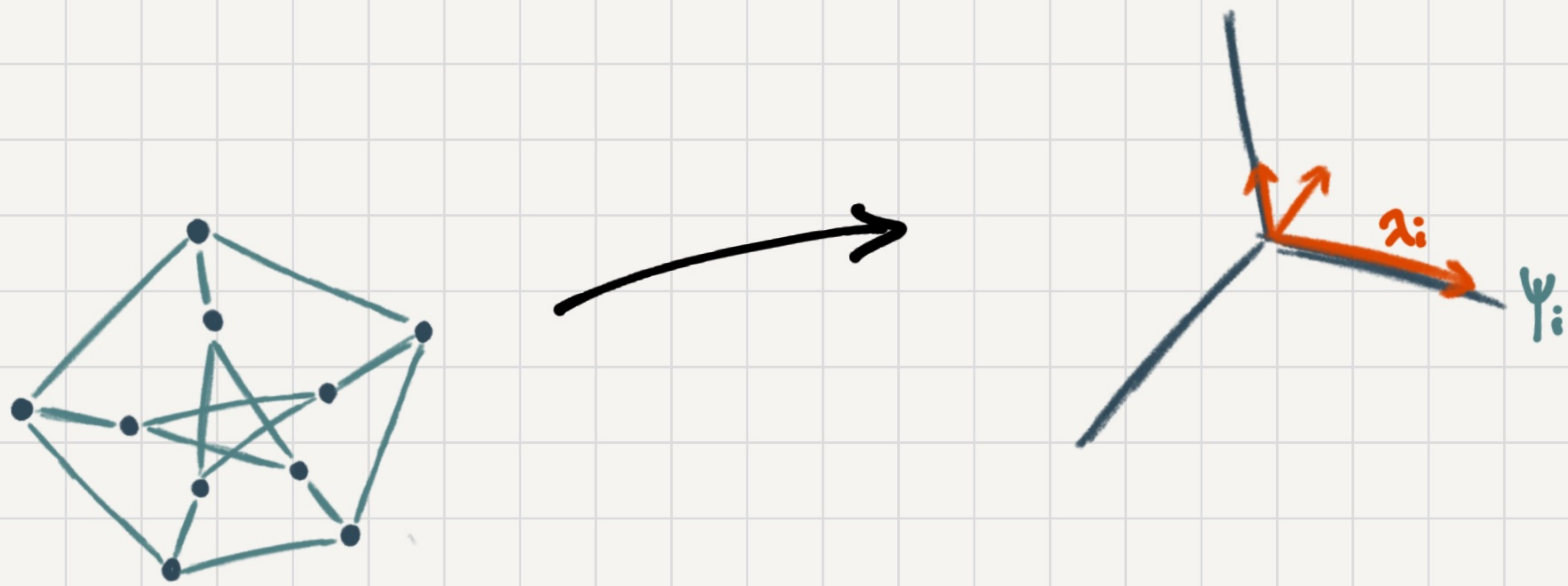
  50% end-of-semester presentation

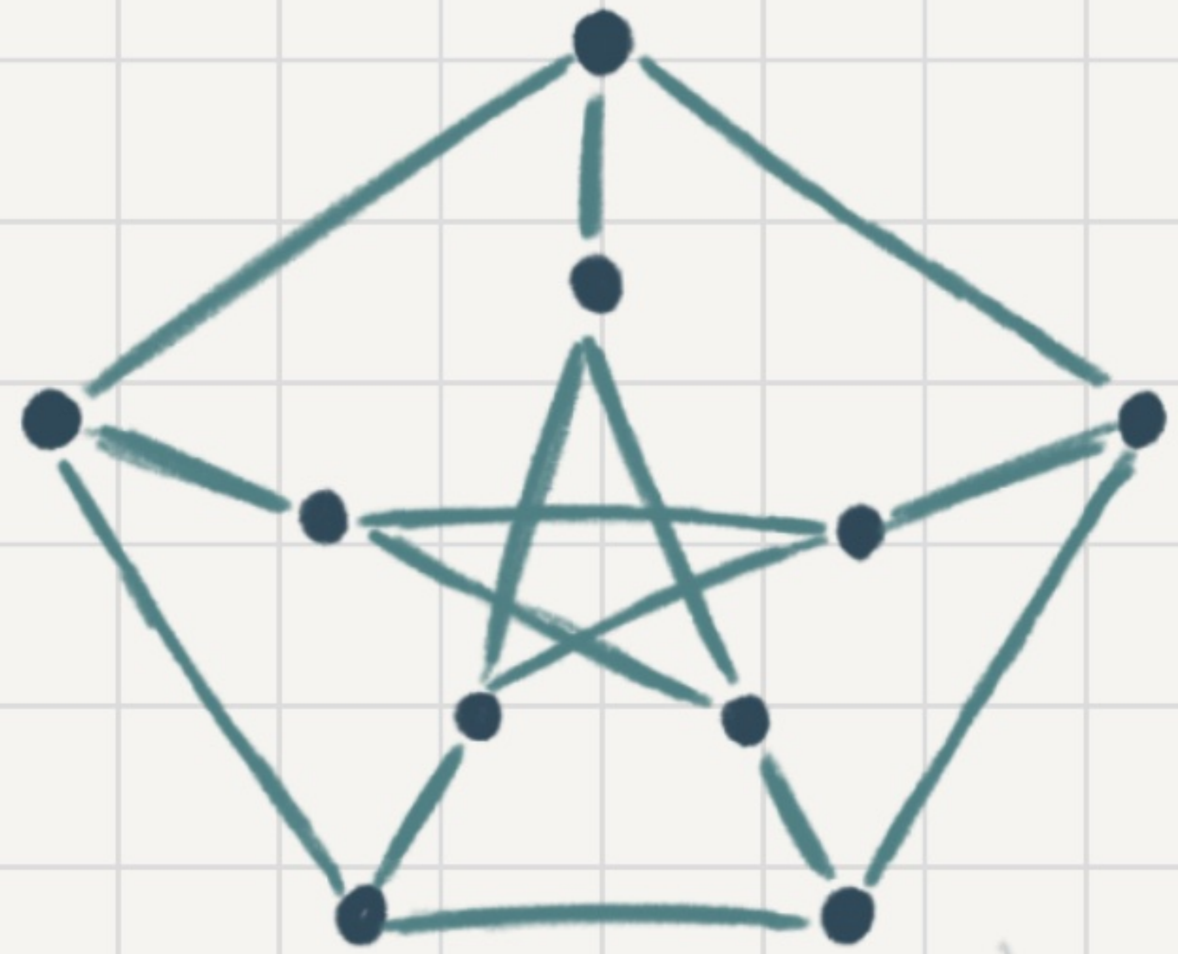
  ~50 minutes, book chapters we won't cover & papers.

* Main book: Lectures on the combinatorics of Free Probability by Nica & Speicher

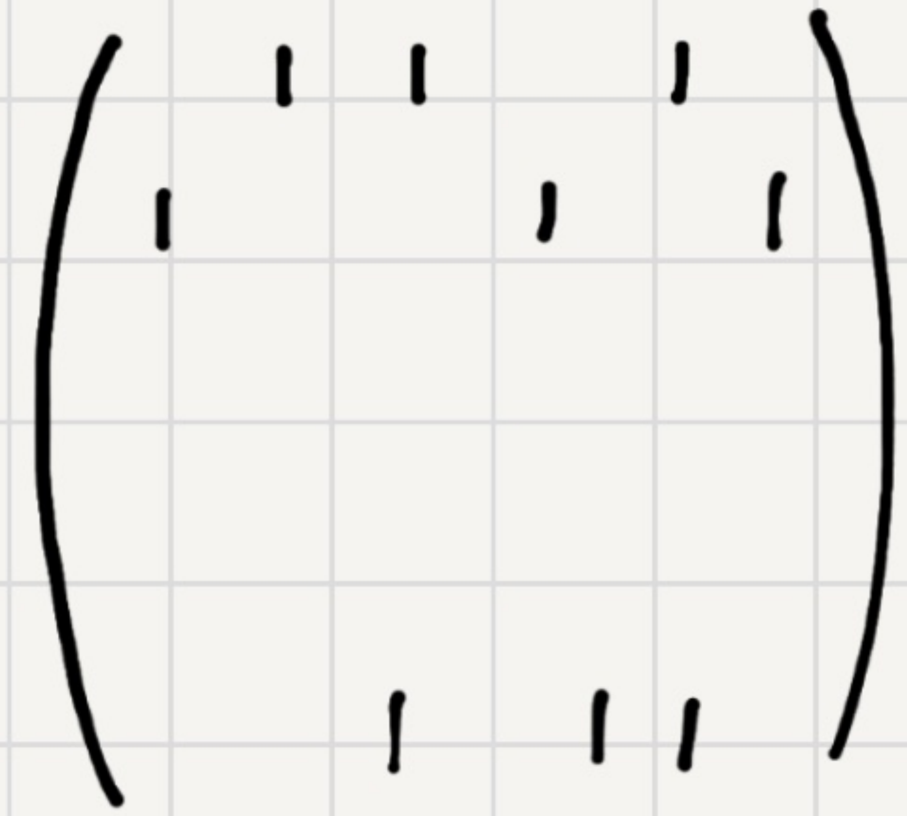
spectral

Expanders Graphs
101

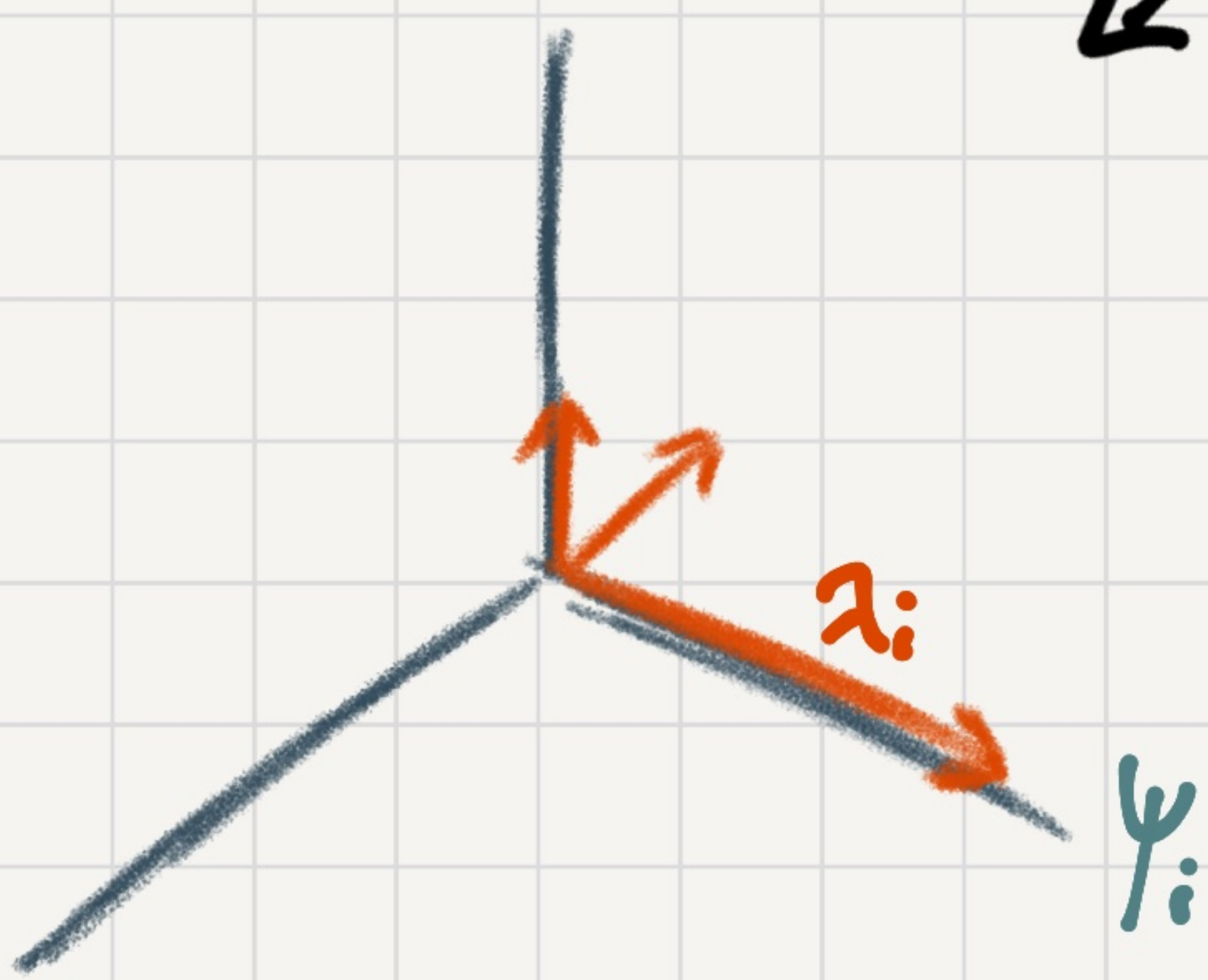




G



A_G

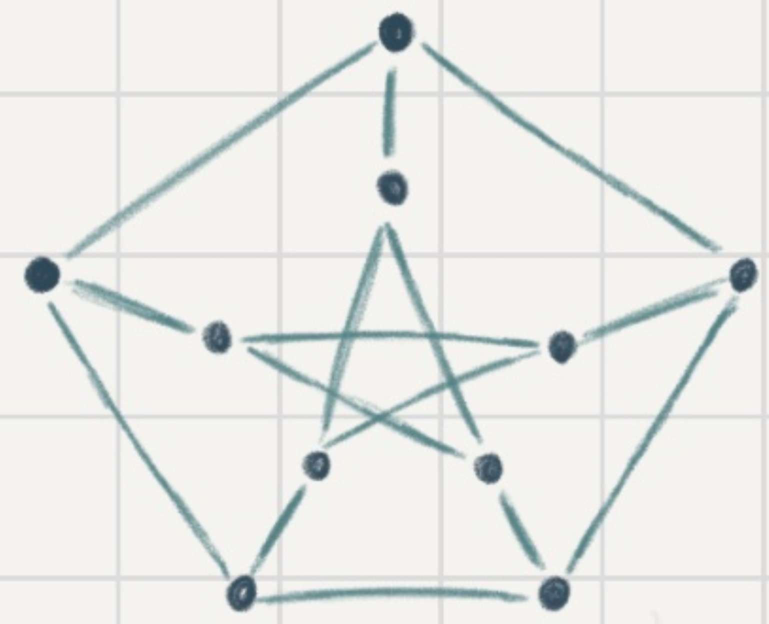


eigenvalue

$$A_G = \sum \lambda_i \psi_i \psi_i^T$$

eigenvector

$$\operatorname{Spec} G = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|}\}$$



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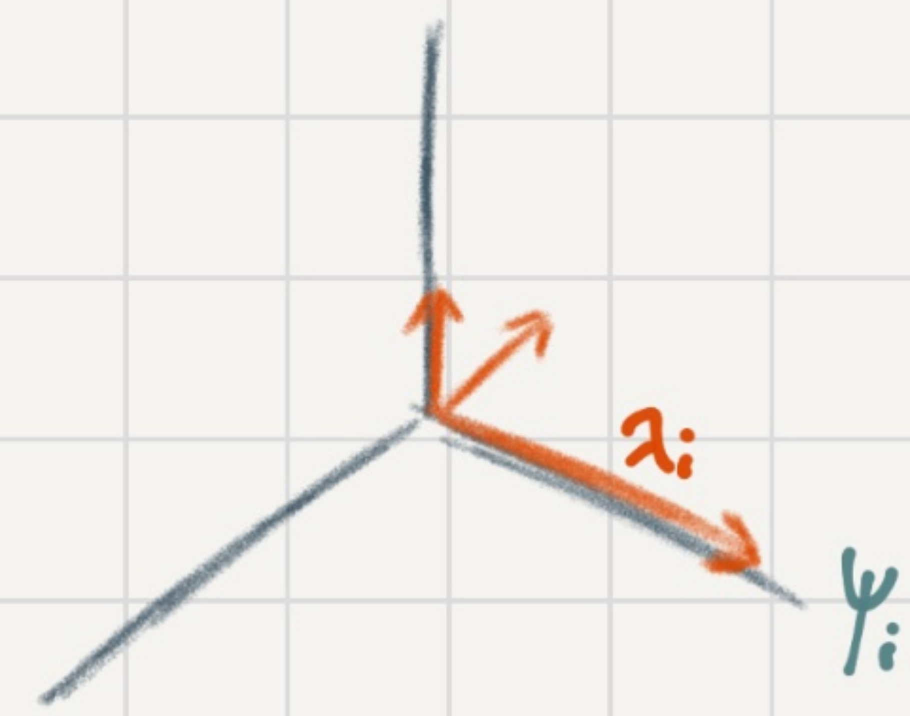

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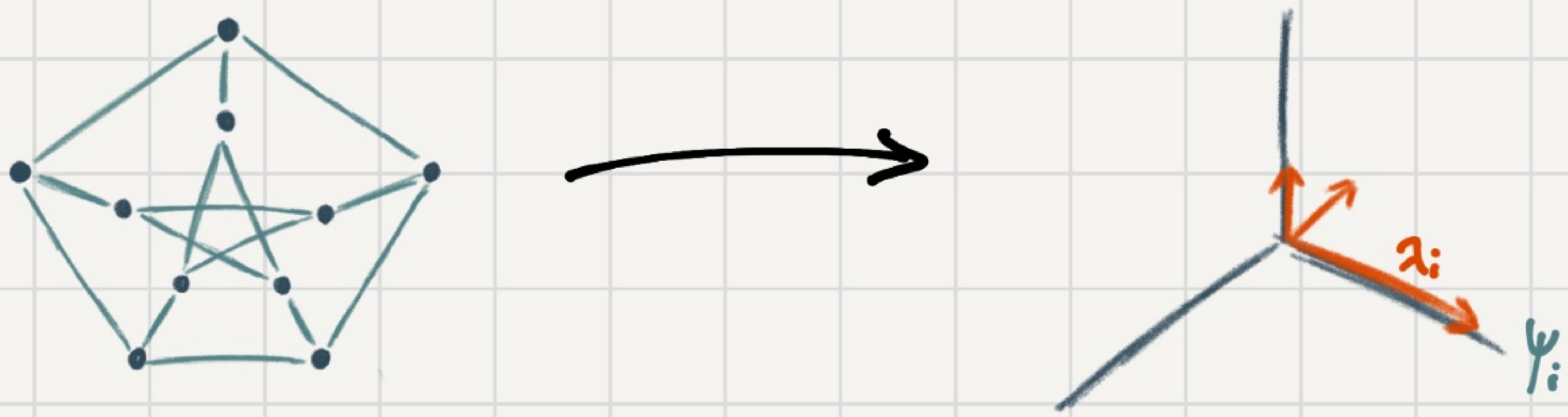
G d-regular $\iff$ $\lambda_1 = d$



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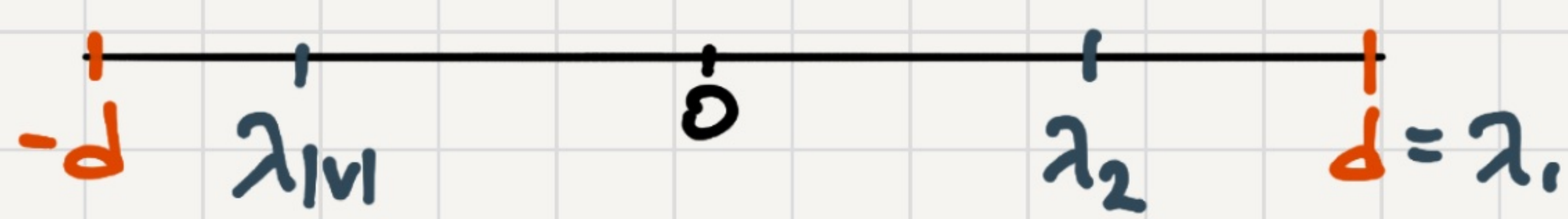

<u>Def.</u> $\lambda_G \triangleq \max\{\lambda_2, |\lambda_{|V|}|\}$ is called the **spectral expansion** of G.

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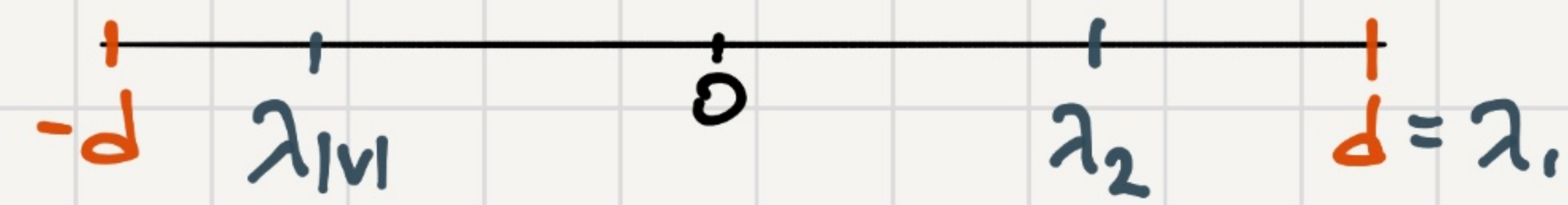

λ_G small $\Longrightarrow$ great things

Expander Chernoff bound

Random walks mix quickly

Expander mixing lemma

Thm [AB '91].

$\forall \varepsilon > 0$ only finitely many d-regular graphs G satisfy

$$\lambda_G < 2\sqrt{d-1} - \varepsilon$$

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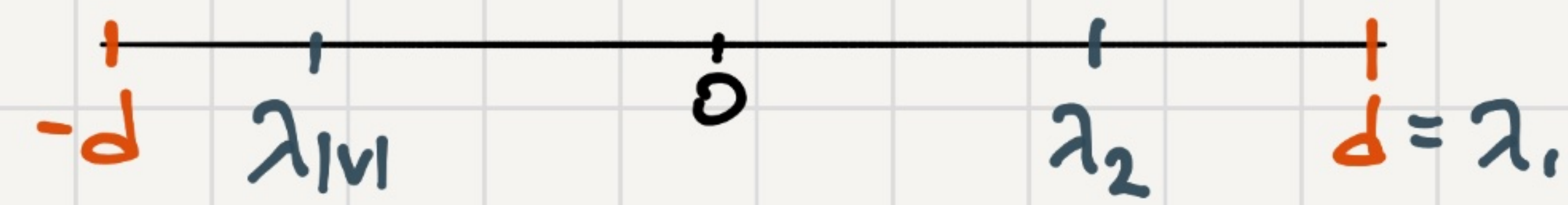


Thm [AB '91].

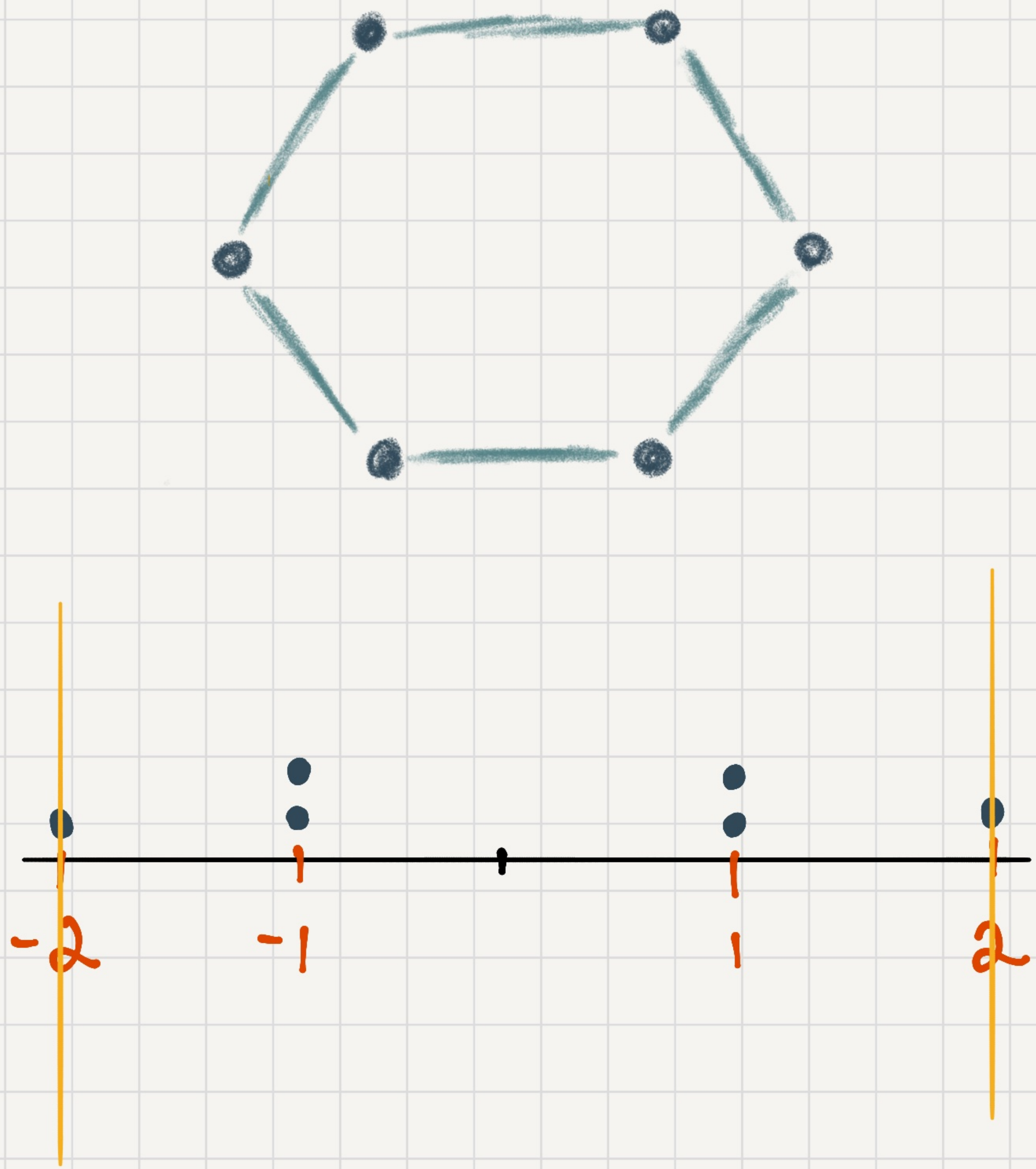
$\forall \varepsilon > 0$ only finitely many d-regular graphs G satisfy

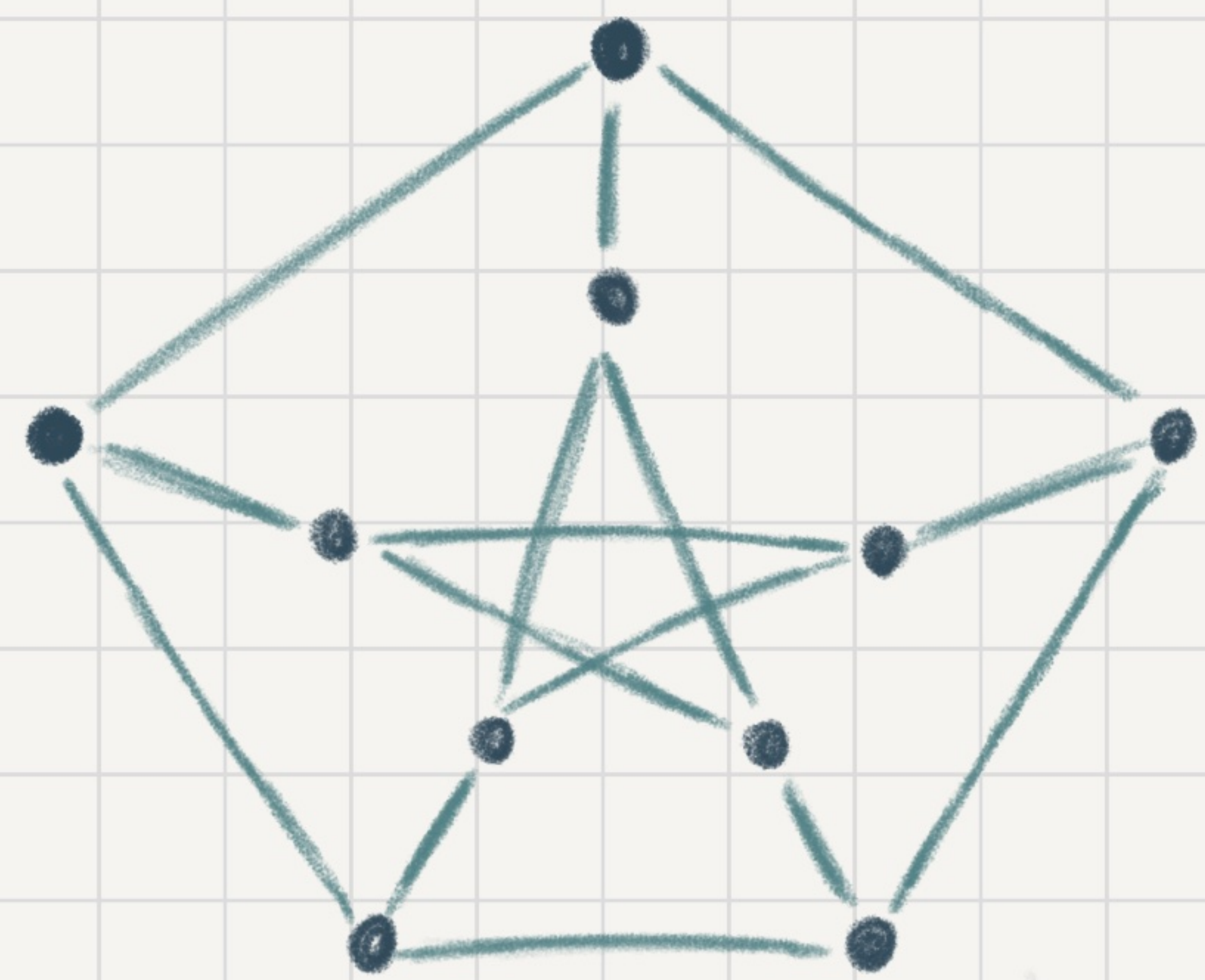
$$\lambda_G < 2\sqrt{d-1} - \varepsilon$$

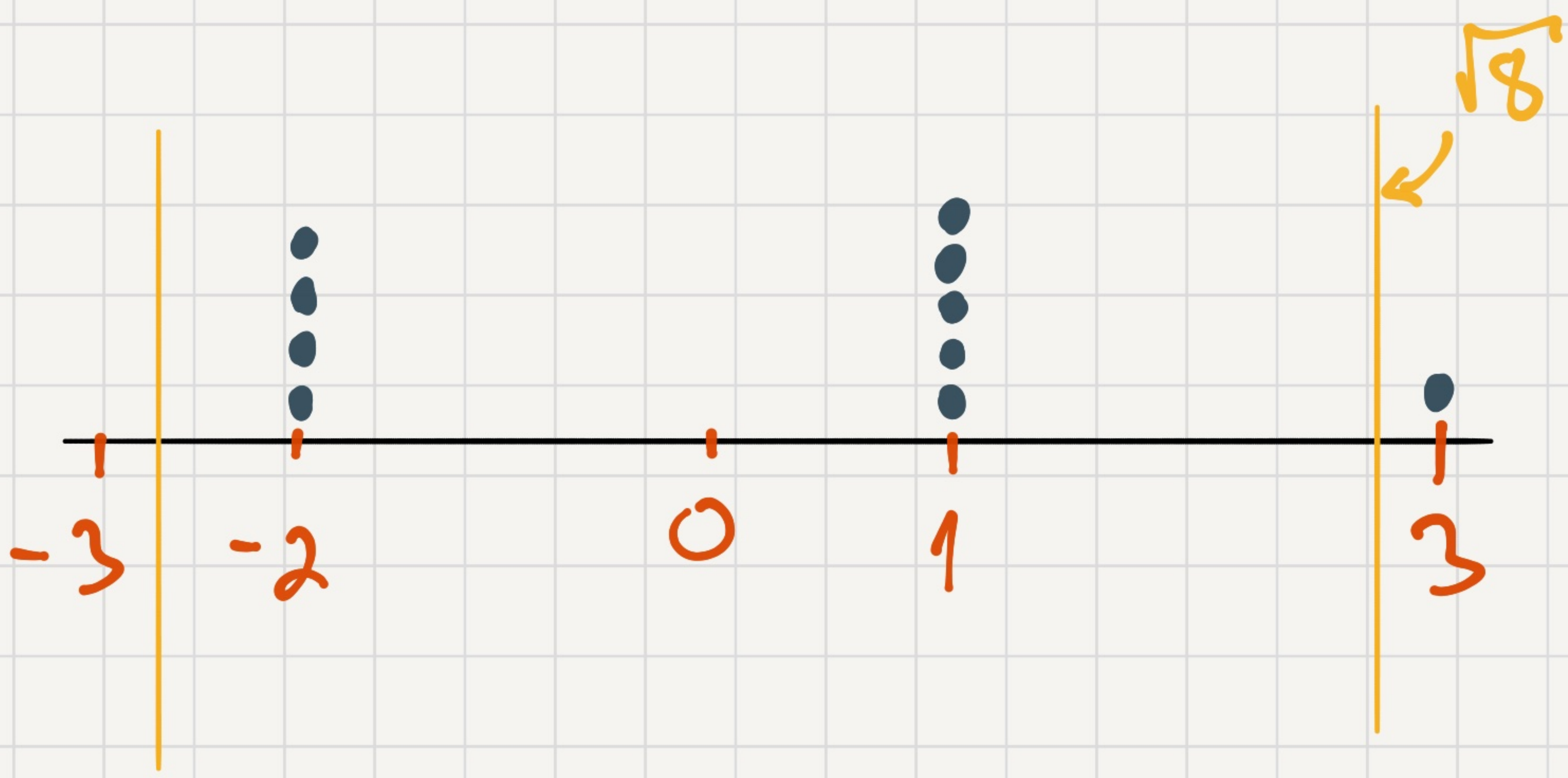
Def.

A d-regular graph G is Ramanujan if

$$\lambda_G \leq 2\sqrt{d-1}.$$


-2
-1
1
2




√8
-3
-2
0
1
3

A typical d-regular graph is known to be nearly Ramanujan $(2\sqrt{d-1}+o(1))$. [F'08, ...]

How do we construct good expanders?

Number/group theory

[LPS'88, Mar'88]

Ramanujan!

Combinatorics & linear algebra

ZigZag [RVW'02...]

Lifting [BL'06...]

⋮

Analytic machinery (~ free probability)

$\forall d$ [MSS'15...]

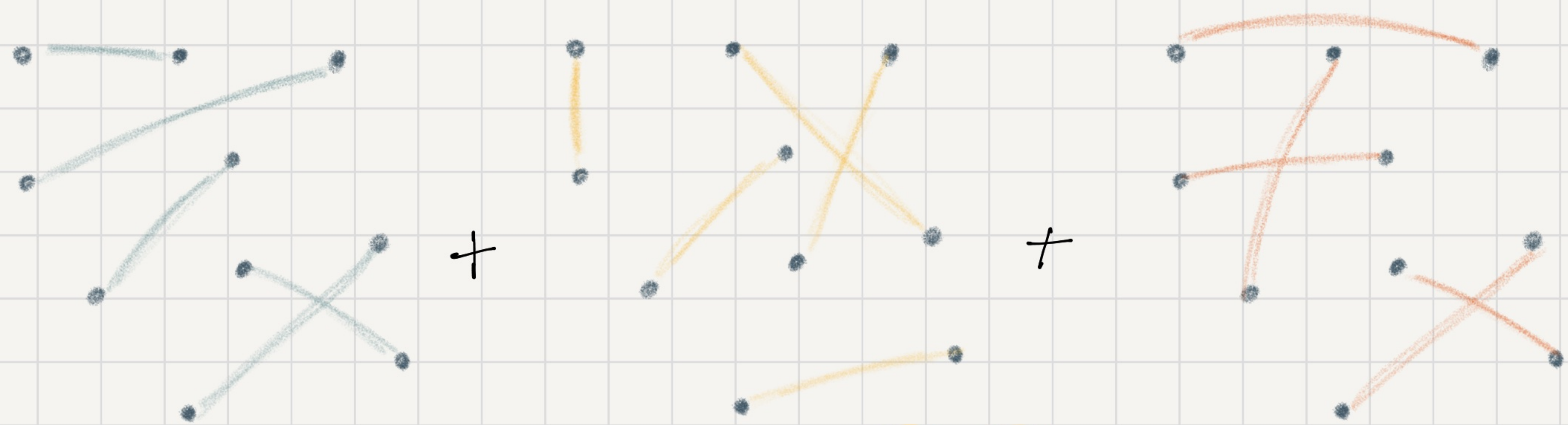
One-sided / Bipartite Ramanujan:

$\lambda_2 \leq 2\sqrt{d-1}$

Marcus-Spielman-Srivastava's idea

The MSS idea.

Take the (multiset) union of d perfect matchings.



a fixed perfect matching

$$A_G = P_1^T M P_1 + P_2^T M P_2 + P_3^T M P_3$$

permutation matrices

Question

Given two $\overbrace{\text{graphs}}^{\text{undirected}}$ G & H with adj matrices A_G, A_H,

what can we say about $\operatorname{Spec}(A_{G\cup H})$?

$A_G + A_H$

Issue Not a function of Spec G & Spec H.

Eigenvectors matter...

Idea "Rotate" one graph:

$$A_G + Q^T A_H Q$$

"random" enough so to decorrelate the evectors of G & H

Idea. "Rotate" one graph:

$$A_G + Q^T A_H Q$$

"random" enough so to decorrelate the evectors of G & H

Very random:
Haar measure

Not a graph ☹

Somewhat random:
random Permutation

A graph! ☺

MSS's Quadrature

essentially

$$\mathbb{E}_{Q}\, \chi_x \left(A_G + Q^T A_H Q \right) = \mathbb{E}_{P}\, \chi_x \left(A_G + P^T A_H P \right)$$

what does it mean?

Does there exist a good P?

Yes! MSS's interlacing

The question remains:

Can we deduce the spectrum of $A + Q^T B Q$ from Spec A & Spec B? How?

Yes! Essentially, we can write

$$\mathbb{E}_{Q}\, \chi_x(A + Q^T B Q) = \chi_x(A) \boxplus \chi_x(B)$$

additive free convolution

This is a finite manifestation of free prob.

Consider the question more abstractly:

If a & b are "free", what is the right formula for computing the mixed moments from the marginals?

E.g.,
$$\mathbb{E}[ab] = \mathbb{E}[a]\,\mathbb{E}[b]$$
$$\mathbb{E}[aba] = \mathbb{E}[a^2]\,\mathbb{E}[b]$$

but

$$\mathbb{E}[abab] = \mathbb{E}[a^2]\,\mathbb{E}[b]^2 + \mathbb{E}[a]^2\,\mathbb{E}[b^2] - \mathbb{E}[a]^2\,\mathbb{E}[b]^2$$
$$\neq \mathbb{E}[a^2]\,\mathbb{E}[b^2] \;!$$

Many questions:

* How do we define "freeness"? Soon...
* Is the new formula for mixed moments unique? Yes! (essentially)
* How come we haven't encountered freeness before? nc & inf-dim
* Is the formula useful - possible to actually use? yes! using analytic machinery
* Conjugating with Haar frees A from B? Yes!
* Quadrature
* Interlacing.

last three lecture

Plan for the Course

Plan for the course.

- Non-commutative probability spaces ← abstract algebraic
- Measure theory & complex analysis 101 $\int_X f\,d\mu$
- A case study → semicircular element, Cauchy trasform & Stieltjes inversion formula
- Free independence
- CLT
- Combinatorics of non-crossing partitions
- free additive convolution

last 3 weeks
- finite free probability
- application (MSS's Ramanujan graphs)