Multiplicative functions in NC

Following Nica-Speicher Chapter 10
(still)

Def. Let $(\alpha_n)_{n\geq 1}$, $\alpha_n \in \mathbb{C}$. Define a family of functions

$$\forall n \geq 1 \qquad F_n \in A(NC(n))$$

$F_n : NC(n) \times NC(n) \longrightarrow \mathbb{C}$
$F_n(\pi,\sigma) \neq 0 \Rightarrow \pi \leq \sigma$

As follows: Let $\pi \leq \sigma$ in $NC(n)$. Let

$$[\pi,\sigma] \cong NC(1)^{k_1} \times \cdots \times NC(n)^{k_n}$$

be the canonical factorization of $[\pi,\sigma]$. Then,

$$F_n(\pi,\sigma) = \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_n^{k_n}$$

$(F_n)_{n\geq 1}$ is called the multiplicative family of functions on $NC^{(2)}$ determined by the sequence $(\alpha_n)_{n\geq 1}$.

A sequence of functions $\left(F_n \in A(NC(n))\right)_{n\geq 1}$ is said to be <u>multiplicative</u> if it arises for a sequence $(\alpha_n)_{n\geq 1}$ as in the above definition.

<u>The one-variable version</u>

<u>Def.</u> Given $(\alpha_n)_{n\geq 1}$, $\alpha_n \in \mathbb{C}$, define a family of functions

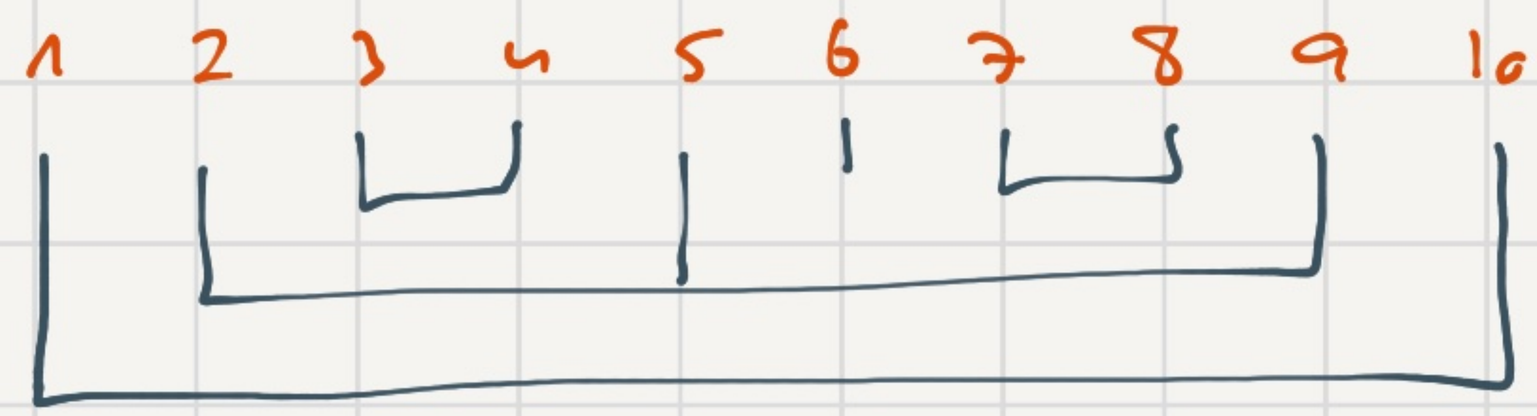
$$\forall n \geq 1 \qquad f_n : NC(n) \to \mathbb{C}$$

as follows: If $\pi = \{V_1, \dots, V_r\} \in NC(n)$ then

$$f_n(\pi) = \alpha_{|V_1|} \cdots \alpha_{|V_r|}$$

Example.

$$f_{10}\left(\quad \right) = \alpha_1 \alpha_2^3 \alpha_3$$



Notation. Given $(\alpha_n)_{n\geq 1}$, $\alpha_n \in \mathbb{C}$ and $\pi \in NC(n)$ we denote

$$\alpha_\pi = f_n(\pi)$$

where (f_n) is the multiplicative family of functions on NC determined by $(\alpha_n)_{n\geq 1}$. We refer to $(\alpha_\pi)_{\pi \in NC}$ as the multiplicative extension of $(\alpha_n)_{n\geq 1}$

$\bigcup_{n\geq 1} NC(n)$

Proposition Let $(f_n)_{n\geq 1}$ be a multiplicative family on NC, and let $(F_n)_{n\geq 1}$ be a multiplicative family on $NC^{(2)}$. Then the family $(f_n * F_n)_{n\geq 1}$ is multiplicative on NC.

pf. Let $g_n \triangleq f_n * F_n$ and

$$\beta_n \triangleq g_n(1_n) = \sum_{\tau \in NC(n)} f_n(\tau) F_n(\tau, 1_n)$$

Fix $n \geq 1$ & $\pi = \{V_1, \ldots, V_r\} \in NC(n)$. We want to prove

$$g(\pi) = \beta_{|V_1|} \cdots \beta_{|V_r|}$$

Now, $\forall \tau \leq \pi$ we have

The image of $\tau|_{V_1}$ under the order-preserving identification of V_1 with $[|V_1|]$

1) $$F_n(\tau, \pi) = F_{|V_1|}(\tau_1, 1_{|V_1|}) \cdots F_{|V_r|}(\tau_r, 1_{|V_r|}).$$

2) $$f_n(\pi) = f_{|V_1|}(\tau_1) \cdots f_{|V_r|}(\tau_r).$$

Thus,

$$g_n(\pi) = \sum_{\tau \leq \pi} f_n(\tau) F_n(\tau, \pi)$$

$$= \sum_{\substack{\tau_1, \ldots, \tau_r : \\ (\tau_i \in NC(|V_i|))}} f_{|V_1|}(\tau_1) F_{|V_1|}(\tau_1, 1_{|V_1|}) \cdots f_{|V_r|}(\tau_r) F_{|V_r|}(\tau_r, 1_{|V_r|})$$

$$= \prod_{k=1}^{r} \overbrace{\sum_{\tau_k \in NC(|V_k|)} f_{|V_k|}(\tau_k) F_{|V_k|}(\tau_k, 1_{|V_k|})}^{\beta_k}$$

$\blacksquare$

Remarks.

$F_n : A(NC(n)) \to \mathbb{C}$

* A similar proof can be used to show that if $(F_n)_{n\geq 1}$ & $(G_n)_{n\geq 1}$ are multiplicative then so is $(F_n * G_n)_{n\geq 1}$.

$G_n : A(NC(n)) \to \mathbb{C}$

* Further, it can be shown that $F_n * G_n = G_n * F_n$.

Functional equation for convolution with

$$\mu_n$$

Theorem. Let $(f_n), (g_n)$ be multiplicative families on NC s.t.

$$g_n = f_n * \mu_n .$$

$\iff f_n = g_n * \zeta_n$

Let $(\alpha_n), (\beta_n)$ be the corresponding sequences, and

$\alpha_n = f(1_n)$
$\beta_n = g(1_n)$

consider the power series

$$F(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n \qquad G(z) = 1 + \sum_{n=1}^{\infty} \beta_n z^n .$$

Then, F, G satisfy the functional equations

$$G(z \cdot F(z)) = F(z) \qquad \& \qquad F\left(\frac{z}{G(z)}\right) = G(z)$$

Pf. As $f_n = g_n * \zeta_n$ we have $\alpha_n = f_n(1_n) = \sum_{\pi \in NC(n)} g_n(\pi)$

Let $V_1 \in \pi$ s.t. $1 \in V_1$. Then,

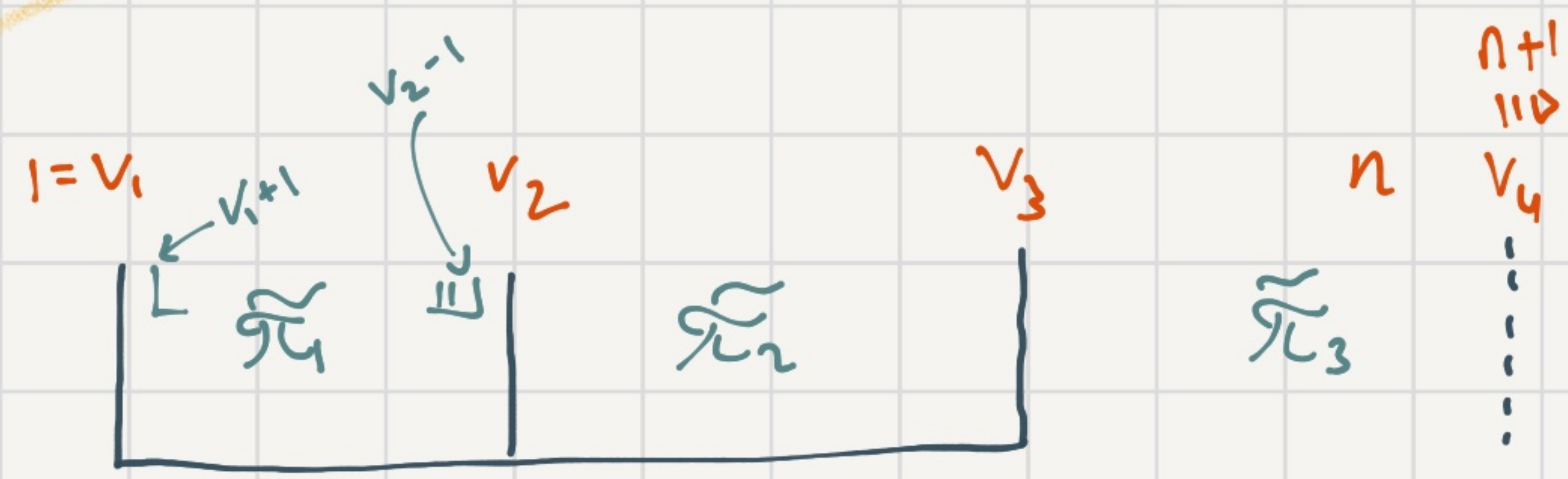
$$\alpha_n = \sum_{s=1}^{n} \sum_{V_1 : |V_1| = s} \sum_{\substack{\pi \in NC(n): \\ 1 \in V_1 \in \pi}} g_n(\pi)$$

Write $V_1 = \{1 = v_1 < v_2 < \cdots < v_s\}$. Then as $\pi \in NC$, it takes the form

using the convention that $v_{s+1} \triangleq n+1$

$$\pi = V_1 \cup \tilde{\pi}_1 \cup \cdots \cup \tilde{\pi}_s$$

where $\tilde{\pi}_j$ is a non-crossing partition of $\{v_j + 1, \ldots, v_{j+1} - 1\}$



$$V_1 = \{v_1, v_2, v_3\}$$

For $j=1,\dots,s$ let $i_j = V_{j+1}-V_j-1$, and idenify $\tilde{\pi}_j$ with an element of $NC(i_j)$

We'll take care of j-s s.t. $i_j=0$ – don't worry

As g is multiplicative $g_n(\pi) = \beta_s\, g_{i_1}(\tilde{\pi}_1)\cdots g_{i_s}(\tilde{\pi}_s)$ where $g_{i_j}(\tilde{\pi}_{i_j}) \overset{\Delta}{=} 1$ when $i_j=0$. Thus,

$$\alpha_n = \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s\in\mathbb{N} \text{ s.t.}\\ i_1+\dots+i_s=n-s}} \underbrace{\sum_{\substack{\pi = V_1\cup\tilde{\pi}_1\cup\dots\cup\tilde{\pi}_s:\\ \tilde{\pi}_j\in NC(i_j)}} \beta_s\, g_{i_1}(\tilde{\pi}_1)\cdots g_{i_s}(\tilde{\pi}_s)}_{\beta_s\cdot\underbrace{\left(\sum_{\tilde{\pi}_1\in NC(i_1)} g_{i_1}(\tilde{\pi}_1)\right)}_{\alpha_{i_1}}\cdots\underbrace{\left(\sum_{\tilde{\pi}_s\in NC(i_s)} g_{i_s}(\tilde{\pi}_s)\right)}_{\alpha_{i_s}}}$$

$$= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s\in\mathbb{N} \text{ s.t.}\\ i_1+\dots+i_s=n-s}} \beta_s\,\alpha_{i_1}\cdots\alpha_{i_s}.$$

Now,

$$F(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n$$

$$= 1 + \sum_{n=1}^{\infty} \sum_{s=1}^{n} \sum_{i_1 + \cdots + i_s = n-s} (\beta_s z^s)(\alpha_{i_1} z^{i_1}) \cdots (\alpha_{i_s} z^{i_s})$$

$$= 1 + \sum_{s=1}^{\infty} \beta_s z^s \underbrace{\sum_{i_1, \ldots, i_s \in N} (\alpha_{i_1} z^{i_1}) \cdots (\alpha_{i_s} z^{i_s})}_{\underbrace{\left(\sum_{i_1=0}^{\infty} \alpha_{i_1} z^{i_1}\right) \cdots \left(\sum_{i_s=0}^{\infty} \alpha_{i_s} z^{i_s}\right)}_{\left(\sum_{i=0}^{n} \alpha_i z^i\right)^s = F(z)^s}}$$

$$= 1 + \sum \beta_s (z \cdot F(z))^s$$

$$= G(z \cdot F(z)).$$

To get the other functional equation, put $w = zF(z)$. Then

$$G(w) \underset{\text{Def of } w}{=} G(zF(z)) \underset{\text{What we just proved}}{=} F(z) \underset{\text{Def of } w}{=} \frac{w}{z}$$

Thus,

$$G(w) \overset{*}{=} F(z) \overset{**}{=} F\left(\frac{w}{G(w)}\right)$$

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