Divisors and Riemann-Roch Spaces

Unit 10

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November 26, 2024

Overview

Prime divisors

Definition 1

Let F/K be a function field. A prime divisor $\mathfrak{p}$ of F/K is a congruence class of places of F/K.

We denote

$$\mathbb{P} = \mathbb{P}_{F/K} = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime divisor of } F/K\} .$$

So

$$\mathfrak{p} = [\varphi] \quad \leftrightarrow \quad \mathcal{O}_{\mathfrak{p}} \quad \leftrightarrow \quad \mathfrak{m}_{\mathfrak{p}} \quad \leftrightarrow \quad [\upsilon]_{\mathfrak{p}}.$$

We proved that all valuations in $[\upsilon]_{\mathfrak{p}}$ are discrete. Further, every $\upsilon_1, \upsilon_2 \in [\upsilon]_{\mathfrak{p}}$ are equal up to a proper normalization. Thus, we pick the unique valuation in $[\upsilon]_{\mathfrak{p}}$ that is onto and denote it by

$$\upsilon_{\mathfrak{p}} : F \to \mathbb{Z} \cup \{\infty\}.$$

Overview

Pseudo divisors

Definition 2

Let F/K be a function field. We denote by $\widetilde{\mathcal{D}}_{F/K}$ the set of formal expressions of the form

$$\sum_{\mathfrak{p} \in \mathbb{P}_{F/K}} n_{\mathfrak{p}} \mathfrak{p},$$

where $n_{\mathfrak{p}} \in \mathbb{Z}$ for all $\mathfrak{p} \in \mathbb{P}_{F/K}$.

Alternatively, $\widetilde{\mathcal{D}}$ is the set of functions $\mathbb{P} \to \mathbb{Z}$.

The elements of $\widetilde{\mathcal{D}}$ are called pseudo divisors.

In some parts of the literature, one writes this in multiplicative form

$$\prod_{\mathfrak{p} \in \mathbb{P}} \mathfrak{p}^{n_{\mathfrak{p}}}$$

to make the resemblance to factorization more explicit.

Pseudo divisors

Note that $\widetilde{\mathcal{D}}$ is a group via the component-wise addition rule

$$\sum_{\mathfrak{p} \in \mathbb{P}} n_{\mathfrak{p}} \mathfrak{p} + \sum_{\mathfrak{p} \in \mathbb{P}} m_{\mathfrak{p}} \mathfrak{p} = \sum_{\mathfrak{p} \in \mathbb{P}} (n_{\mathfrak{p}} + m_{\mathfrak{p}}) \mathfrak{p}.$$

For $\mathfrak{q} \in \mathbb{P}$ we define $\upsilon_{\mathfrak{q}} : \widetilde{\mathcal{D}} \to \mathbb{Z}$ by

$$\upsilon_{\mathfrak{q}} \left(\sum_{\mathfrak{p} \in \mathbb{P}} n_{\mathfrak{p}} \mathfrak{p} \right) = n_{\mathfrak{q}}.$$

Note that $\upsilon_{\mathfrak{q}}$ is a group homomorphism.

Partial order on pseudo divisors

We define a partial order on $\widetilde{\mathcal{D}}$ by

$$\mathfrak{a} \leq \mathfrak{b} \quad \Longleftrightarrow \quad \forall \mathfrak{p} \in \mathbb{P} \quad v_{\mathfrak{p}}(\mathfrak{a}) \leq v_{\mathfrak{p}}(\mathfrak{b}).$$

We define

$$\max(\mathfrak{a}, \mathfrak{b}) = \sum_{\mathfrak{p} \in \mathbb{P}} \max(v_{\mathfrak{p}}(\mathfrak{a}), v_{\mathfrak{p}}(\mathfrak{b}))\mathfrak{p} \in \widetilde{\mathcal{D}},$$

$$\min(\mathfrak{a}, \mathfrak{b}) = \sum_{\mathfrak{p} \in \mathbb{P}} \min(v_{\mathfrak{p}}(\mathfrak{a}), v_{\mathfrak{p}}(\mathfrak{b}))\mathfrak{p} \in \widetilde{\mathcal{D}}.$$

We further note that

$$\mathfrak{a} \leq \mathfrak{b} \quad \Longrightarrow \quad \mathfrak{a} + \mathfrak{c} \leq \mathfrak{b} + \mathfrak{c}.$$

Overview

Divisors

Definition 3 (Divisors)

Let F/K be a function field, and $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_{\mathsf{F/K}}$. An element $\mathfrak{a} \in \widetilde{\mathcal{D}}$ is called a divisor if $v_{\mathfrak{p}}(\mathfrak{a}) = 0$ for all but finitely many $\mathfrak{p} \in \mathbb{P}$. In this case, we say that $v_{\mathfrak{p}}(\mathfrak{a}) = 0$ almost always.

The set of divisors of F/K is denoted by $\mathcal{D} = \mathcal{D}_{\mathsf{F/K}}$.

Note that $\mathcal{D}$ is a subgroup of $\widetilde{\mathcal{D}}$.

Degree of divisors

Definition 4

Let F/K be a function field. The **degree** of a prime divisor $\mathfrak{p} \in \mathbb{P}$, denoted $\deg \mathfrak{p}$, is defined to be $\deg \varphi$ where $\varphi : F \to L \cup \{\infty\}$ is any place that corresponds to $\mathfrak{p}$. That is,

$$\deg \mathfrak{p} = \left[\mathcal{O} \big/ \mathfrak{m} : K \right].$$

We extend this definition to a general divisor $\mathfrak{a} \in \mathcal{D}$ by setting

$$\deg(\mathfrak{a}) = \sum_{\mathfrak{p} \in \mathbb{P}} v_{\mathfrak{p}}(\mathfrak{a}) \deg \mathfrak{p}.$$

Observe that $\deg : \mathcal{D} \to \mathbb{Z}$ is a group homomorphism that preserves the partial order, that is,

$$\mathfrak{a} \leq \mathfrak{b} \implies \deg \mathfrak{a} \leq \deg \mathfrak{b}.$$

Principal divisors

For $x \in F^{\times}$ we define

$$(x) = \sum_{\mathfrak{p} \in \mathbb{P}} v_{\mathfrak{p}}(x)\mathfrak{p}.$$

Recall that a valuation of F/K is trivial on K and is non-trivial on F. Thus,

$$x \in K^{\times} \iff (x) = 0.$$

In the rational function field $K(x)/K$,

$$(x) = \mathfrak{p}_0 - \mathfrak{p}_\infty,$$

$$\left(\frac{x^3}{x-1}\right) = 3\mathfrak{p}_0 - \mathfrak{p}_1 - 2\mathfrak{p}_\infty.$$

Principal divisors

In our ongoing example $y^2 = x^3 - x$, for characteristic $\neq 2$,

$$\begin{aligned}(x) &= 2\mathfrak{p}_{0,0} - 2\mathfrak{p}_\infty,\\ (y) &= \mathfrak{p}_{0,0} + \mathfrak{p}_{1,0} + \mathfrak{p}_{-1,0} - 3\mathfrak{p}_\infty,\\ \left(\frac{x}{y}\right) &= \mathfrak{p}_{0,0} + \mathfrak{p}_\infty - \mathfrak{p}_{1,0} - \mathfrak{p}_{-1,0},\end{aligned}$$

whereas in characteristic 2,

$$(y) = \mathfrak{p}_{0,0} + 2\mathfrak{p}_{1,0} - 3\mathfrak{p}_\infty.$$

Overview

Riemann-Roch spaces

Let $S \subseteq \mathbb{P}$. For $\mathfrak{a} \in \widetilde{\mathcal{D}}$ we define $\mathfrak{a}_S \in \widetilde{\mathcal{D}}$ by

$$v_{\mathfrak{p}}(\mathfrak{a}_S) = \begin{cases} v_{\mathfrak{p}}(\mathfrak{a}), & \mathfrak{p} \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Further define

$$\begin{aligned} \mathcal{L}(\mathfrak{a}, S) &= \{x \in \mathsf{F} \mid \forall \mathfrak{p} \in S \ \ v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0\} \\ &= \left\{x \in \mathsf{F}^{\times} \mid (x)_S + \mathfrak{a}_S \geq 0\right\} \cup \{0\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(\mathfrak{a}) = \mathcal{L}(\mathfrak{a}, \mathbb{P}) &= \{x \in \mathsf{F} \mid \forall \mathfrak{p} \in \mathbb{P} \ \ v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(\mathfrak{a}) \geq 0\} \\ &= \left\{x \in \mathsf{F}^{\times} \mid (x) + \mathfrak{a} \geq 0\right\} \cup \{0\}. \end{aligned}$$

Definition 5

If $\mathfrak{a} \in \mathcal{D}$ we call $\mathcal{L}(\mathfrak{a})$ a Riemann-Roch space.

Riemann-Roch spaces

Claim 6

Let F/K be a function field. Let $\mathfrak{a} \in \widetilde{\mathcal{D}}$ and $S \subseteq \mathbb{P}$. Then, $\mathcal{L}(\mathfrak{a}, S)$ is a K-vector space, a subspace of F.

Proof.

$0 \in \mathcal{L}(\mathfrak{a}, S)$ by definition.

Now, if $x, y \in \mathcal{L}(\mathfrak{a}, S)$ then

$$\forall \mathfrak{p} \in \mathbb{P} \quad \upsilon_{\mathfrak{p}}(x), \upsilon_{\mathfrak{p}}(y) \geq -\upsilon_{\mathfrak{p}}(\mathfrak{a}),$$

and so

$$\forall \mathfrak{p} \in \mathbb{P} \quad \upsilon_{\mathfrak{p}}(x+y) \geq \min(\upsilon_{\mathfrak{p}}(x), \upsilon_{\mathfrak{p}}(y)) \geq -\upsilon_{\mathfrak{p}}(\mathfrak{a}).$$

Thus, $x + y \in \mathcal{L}(\mathfrak{a}, S)$.

Since $\upsilon_{\mathfrak{p}}(x) = 0$ for every $x \in K^{\times}$, $\mathcal{L}(\mathfrak{a}, S)$ is closed under multiplication by a scalar. □

Riemann-Roch spaces

Definition 7

For a divisor $\mathfrak{a} \in \mathcal{D}$ we denote

$$\dim \mathfrak{a} = \dim_K \mathcal{L}(\mathfrak{a}).$$

Claim 8

For every $\mathfrak{a} \in \widetilde{\mathcal{D}}$ and $S_1, S_2 \subseteq \mathbb{P}$,

$$S_1 \subseteq S_2 \quad \Longrightarrow \quad \mathcal{L}(\mathfrak{a}, S_2) \subseteq \mathcal{L}(\mathfrak{a}, S_1).$$

In particular, $\mathcal{L}(\mathfrak{a}) \subseteq \mathcal{L}(\mathfrak{a}, S)$.

The proof is straightforward by the definitions.

Riemann-Roch spaces

Note that for every $x \in F^\times$, the map $F \to F$ mapping $y \mapsto xy$ is K-linear.

Claim 9

For every $\mathfrak{a} \in \widetilde{\mathcal{D}}$, $S \subseteq \mathbb{P}$, and $x \in F^\times$,

$$x\mathcal{L}(\mathfrak{a}, S) = \mathcal{L}(\mathfrak{a} - (x), S).$$

Proof.

$$\begin{aligned}
y \in x\mathcal{L}(\mathfrak{a}, S) \quad &\iff \quad \frac{y}{x} \in \mathcal{L}(\mathfrak{a}, S) \\
&\iff \quad \forall \mathfrak{p} \in S \;\; v_\mathfrak{p}(y/x) + v_\mathfrak{p}(\mathfrak{a}) \geq 0 \\
&\iff \quad \forall \mathfrak{p} \in S \;\; v_\mathfrak{p}(y) - v_\mathfrak{p}(x) + v_\mathfrak{p}(\mathfrak{a}) \geq 0 \\
&\iff \quad \forall \mathfrak{p} \in S \;\; v_\mathfrak{p}(y) - v_\mathfrak{p}((x)) + v_\mathfrak{p}(\mathfrak{a}) \geq 0 \\
&\iff \quad \forall \mathfrak{p} \in S \;\; v_\mathfrak{p}(y) + v_\mathfrak{p}(\mathfrak{a} - (x)) \geq 0 \\
&\iff \quad y \in \mathcal{L}(\mathfrak{a} - (x), S).
\end{aligned}$$

□

Riemann-Roch spaces

Lemma 10

Let $S \subseteq \mathbb{P}$ *finite and* $\mathfrak{a}, \mathfrak{b} \in \widetilde{\mathcal{D}}$ *s.t.* $\mathfrak{a} \le \mathfrak{b}$ *(* $\Longrightarrow \mathcal{L}(\mathfrak{a}, S) \subseteq \mathcal{L}(\mathfrak{b}, S)$*). Then,*

$$\dim_K \mathcal{L}(\mathfrak{b}, S) \Big/ \mathcal{L}(\mathfrak{a}, S) = \deg \mathfrak{b}_S - \deg \mathfrak{a}_S .$$

Proof.

First, we may assume that $\mathfrak{a} = \mathfrak{a}_S$, $\mathfrak{b} = \mathfrak{b}_S$ since $\mathcal{L}(\mathfrak{a}, S) = \mathcal{L}(\mathfrak{a}_S, S)$.

It suffices to consider the case $\mathfrak{b} = \mathfrak{a} + \mathfrak{p}$ for some prime divisor $\mathfrak{p} \in \mathbb{P}$. To see this, recall that by the third isomorphism theorem, if $V_1 \subseteq V_2 \subseteq V_3$ are K-vector spaces then

$$\dim_K V_3 \Big/ V_1 = \dim_K V_3 \Big/ V_2 + \dim_K V_2 \Big/ V_1 .$$

Thus, it suffices to prove that for $\mathfrak{p} \in S$ s.t. $\mathfrak{a} \le \mathfrak{a} + \mathfrak{p} \le \mathfrak{b}$ we have

$$\dim_K \mathcal{L}(\mathfrak{a} + \mathfrak{p}, S) \Big/ \mathcal{L}(\mathfrak{a}, S) = \deg \mathfrak{p} .$$

Riemann-Roch spaces

Proof.

By the weak approximation theorem (WAT), $\exists x \in \mathrm{F}$ s.t.

$$\forall \mathfrak{q} \in S \quad v_{\mathfrak{q}}(x) = v_{\mathfrak{q}}(\mathfrak{a} + \mathfrak{p}).$$

Equivalently,

$$(x)_S = (\mathfrak{a} + \mathfrak{p})_S.$$

By Claim 9,

$$x\mathcal{L}(\mathfrak{a} + \mathfrak{p}, S) = \mathcal{L}(\mathfrak{a} + \mathfrak{p} - (x), S) = \mathcal{L}(0, S),$$
$$x\mathcal{L}(\mathfrak{a}, S) = \mathcal{L}(\mathfrak{a} - (x), S) = \mathcal{L}(-\mathfrak{p}, S).$$

Thus, it suffices to prove that

$$\dim_{\mathrm{K}} \mathcal{L}(0, S) \Big/ \mathcal{L}(-\mathfrak{p}, S) = \deg \mathfrak{p}.$$

Proof.

To summarize, we wish to prove

$$\dim_K \mathcal{L}(0,S) \big/ \mathcal{L}(-\mathfrak{p},S) = \deg \mathfrak{p}.$$

Note that as $\mathfrak{p} \in S$,

$$\mathcal{L}(0,S) \subseteq \mathcal{L}(0,\{\mathfrak{p}\}) = \mathcal{O}_{\mathfrak{p}}.$$

Denote $F_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}} \big/ \mathfrak{m}_{\mathfrak{p}}$. We will show that restricting the projection map $\mathcal{O}_{\mathfrak{p}} \to F_{\mathfrak{p}}$ to $\mathcal{L}(0,S)$, namely,

$$\varphi : \mathcal{L}(0,S) \to F_{\mathfrak{p}}$$
$$x \mapsto x + \mathfrak{m}_{\mathfrak{p}}$$

is onto with $\ker \varphi = \mathcal{L}(-\mathfrak{p},S)$. This will complete the proof as, recall,

$$\deg \mathfrak{p} = [F_{\mathfrak{p}} : K] = \dim_K F_{\mathfrak{p}}.$$

Proof.

We start by proving that

$$\begin{aligned}\varphi : \mathcal{L}(0, S) &\to F_{\mathfrak{p}}\\ x &\mapsto x + \mathfrak{m}_{\mathfrak{p}}\end{aligned}$$

is onto. Take $\bar{x} \in F_{\mathfrak{p}}$ and $x \in \mathcal{O}_{\mathfrak{p}}$ s.t. $\varphi(x) = \bar{x}$. By WAT, $\exists y \in F$ s.t.

$$\begin{aligned}&\upsilon_{\mathfrak{p}}(y - x) > 0,\\ &\upsilon_{\mathfrak{q}}(y) \geq 0 \quad \forall \mathfrak{q} \in S \setminus \{\mathfrak{p}\}.\end{aligned}$$

Thus,

$$\upsilon_{\mathfrak{p}}(y) \geq \min(\upsilon_{\mathfrak{p}}(x), \upsilon_{\mathfrak{p}}(y - x)) \geq 0,$$

and so $y \in \mathcal{L}(0, S)$. As $\varphi(y - x) = 0$,

$$\varphi(y) = \varphi(x) = \bar{x}.$$

φ is therefore onto.

Riemann-Roch spaces

Proof.

$$\varphi : \mathcal{L}(0,S) \to F_{\mathfrak{p}}$$
$$x \mapsto x + \mathfrak{m}_{\mathfrak{p}}$$

We turn to prove that $\ker\varphi = \mathcal{L}(-\mathfrak{p}, S)$.

To see this, take $x \in \mathcal{L}(0,S)$ and note that

$$\varphi(x) = 0 \iff v_{\mathfrak{p}}(x) > 0 \iff x \in \mathcal{L}(-\mathfrak{p}, S).$$

Claim 11

Let $\mathfrak{a} \in \mathcal{D}$, $\mathfrak{a} < 0$. Then, $\mathcal{L}(\mathfrak{a}) = 0$.

Proof.

To prove the claim, recall that for every $x \in \mathsf{F} \setminus \mathsf{K}$ there are valuations υ, υ' of F/K s.t. $\upsilon(x) > 0$ yet $\upsilon'(x) < 0$.

Take $x \in \mathcal{L}(\mathfrak{a})$. Then,

$$\forall \mathfrak{p} \in \mathbb{P} \quad \upsilon_{\mathfrak{p}}(x) \geq -\upsilon_{\mathfrak{p}}(\mathfrak{a}) \geq 0 \quad \Longrightarrow \quad x \in \mathsf{K}.$$

However, $\mathfrak{a} < 0$ and so $\exists \mathfrak{q} \in \mathbb{P}$ s.t. $\upsilon_{\mathfrak{q}}(\mathfrak{a}) < 0$ and so $\upsilon_{\mathfrak{q}}(x) > 0$. whereas $\upsilon_q(\mathsf{K}^\times) = 0$. Thus, $x = 0$.

Claim 12

$\mathcal{L}(0) = \mathsf{K}$.

Proof.

All valuations of F/K are trivial on K and so $\mathsf{K} \subseteq \mathcal{L}(0)$.

On the other hand, if $x \in \mathcal{L}(0)$ then $v_{\mathfrak{p}}(x) \geq 0$ for all $\mathfrak{p} \in \mathbb{P}$, and so $x \in \mathsf{K}$.

Claim 13

Let $\mathfrak{a} \le \mathfrak{b}$ be divisors, and let $S \subseteq \mathbb{P}$ be the set of all prime divisors appearing in $\mathfrak{a}, \mathfrak{b}$. Then,

$$\mathcal{L}(\mathfrak{b}) \cap \mathcal{L}(\mathfrak{a}, S) = \mathcal{L}(\mathfrak{a}).$$

Proof.

Clearly, $\mathcal{L}(\mathfrak{a}) \subseteq \mathcal{L}(\mathfrak{b})$ and $\mathcal{L}(\mathfrak{a}) \subseteq \mathcal{L}(\mathfrak{a}, S)$.

For the other direction, take $x \in \mathcal{L}(\mathfrak{b}) \cap \mathcal{L}(\mathfrak{a}, S)$. Then,

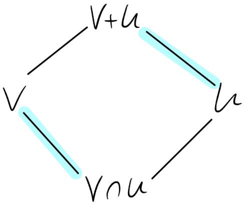
$$\forall \mathfrak{p} \in S \quad \upsilon_{\mathfrak{p}}(x) + \upsilon_{\mathfrak{p}}(\mathfrak{a}) \ge 0.$$

It remains to argue about $\mathfrak{p} \notin S$. But, then $\upsilon_{\mathfrak{p}}(\mathfrak{a}) = \upsilon_{\mathfrak{p}}(\mathfrak{b}) = 0$ and since $x \in \mathcal{L}(\mathfrak{b})$ we have

$$\upsilon_{\mathfrak{p}}(x) + \upsilon_{\mathfrak{p}}(\mathfrak{a}) = \upsilon_{\mathfrak{p}}(x) + 0 = \upsilon_{\mathfrak{p}}(x) + \upsilon_{\mathfrak{p}}(\mathfrak{b}) \ge 0.$$

Let U, V be K-vector spaces. Then, $U+V$ and $U \cap V$ are also K-vector spaces, and

$$(V+U)\Big/U \cong V\Big/V \cap U.$$



Lemma 14

For divisors $\mathfrak{a} \le \mathfrak{b}$,

$$\dim_K \mathcal{L}(\mathfrak{b}) \Big/ \mathcal{L}(\mathfrak{a}) \le \deg \mathfrak{b} - \deg \mathfrak{a}.$$

Proof.

Let $S \subseteq \mathbb{P}$ be the set of all prime divisors appearing in $\mathfrak{a}, \mathfrak{b}$. Note $|S| < \infty$.

By Lemma 10,

$$\dim_K \mathcal{L}(\mathfrak{b}, S) \Big/ \mathcal{L}(\mathfrak{a}, S) = \deg \mathfrak{b}_S - \deg \mathfrak{a}_S.$$

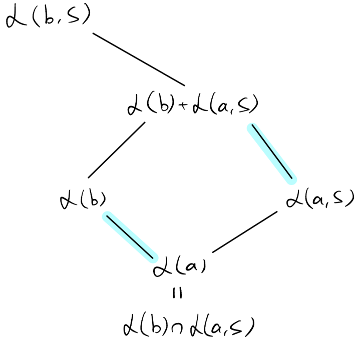
Riemann-Roch spaces

Proof.

$$\dim_K \mathcal{L}(\mathfrak{b}, S) \Big/ \mathcal{L}(\mathfrak{a}, S) = \deg \mathfrak{b}_S - \deg \mathfrak{a}_S.$$

The proof then follows as the diagram shows that

$$\mathcal{L}(\mathfrak{b}) \Big/ \mathcal{L}(\mathfrak{a}) \leq \mathcal{L}(\mathfrak{b}, S) \Big/ \mathcal{L}(\mathfrak{a}, S).$$



Riemann-Roch spaces

We are now in a position to prove that Riemann-Roch spaces are of finite dimension as K-vector spaces.

Corollary 15

For every $\mathfrak{a} \in \mathcal{D}$, $\dim \mathfrak{a} < \infty$.

Proof.

Let $\mathfrak{b} < 0$ a divisor. By Lemma 14,

$$\dim_K \mathcal{L}(\mathfrak{a}) \big/ \mathcal{L}(\min(\mathfrak{a}, \mathfrak{b})) \leq \deg \mathfrak{a} - \deg \min(\mathfrak{a}, \mathfrak{b}).$$

But Claim 11 implies

$$\mathcal{L}(\min(\mathfrak{a}, \mathfrak{b})) \subseteq \mathcal{L}(\mathfrak{b}) = 0.$$

Thus,

$$\dim \mathfrak{a} = \dim_K \mathcal{L}(\mathfrak{a}) \leq \deg \mathfrak{a} - \deg \min(\mathfrak{a}, \mathfrak{b}) < \infty.$$

□

Riemann-Roch spaces

Another corollaries of Lemma 14 is

Corollary 16

For every $\mathfrak{a}, \mathfrak{b} \in \mathcal{D}$*,*

$$\mathfrak{a} \leq \mathfrak{b} \quad \Longrightarrow \quad \deg \mathfrak{a} - \dim \mathfrak{a} \leq \deg \mathfrak{b} - \dim \mathfrak{b}.$$

We will soon prove that

$$\sup_{\mathfrak{a} \in \mathcal{D}_{F/K}} (\deg \mathfrak{a} - \dim \mathfrak{a}) < \infty.$$

This will lead to the definition of the genus of a function field.

Riemann-Roch spaces

Based on Lemma 14 we can strengthen Corollary 15 for non-negative divisors.

Corollary 17

For every $\mathfrak{a} \in \mathcal{D}$, $\mathfrak{a} \geq 0$ *we have*

$$\dim \mathfrak{a} \leq \deg \mathfrak{a} + 1.$$

The proof is left as an exercise.

Example

Consider the rational function field $F = \mathbb{F}_q(x)/\mathbb{F}_q$.

For $r \in \mathbb{N}$, $\mathcal{L}(r\mathfrak{p}_\infty)$ consists of all $f(x) \in \mathbb{F}_q(x)$ s.t.

$$\upsilon_\infty(f(x)) \geq -r,$$
$$\upsilon_\mathfrak{p}(f(x)) \geq 0 \quad \forall \mathfrak{p} \in \mathbb{P} \setminus \{\mathfrak{p}_\infty\}.$$

The second condition implies that $f(x) \in \mathbb{F}_q[x]$.

The first condition then implies $\deg f(x) \leq r$.

Hence, $\mathcal{L}(r\mathfrak{p}_\infty)$ is the $\mathbb{F}_q$-vector space of polynomials of degree $\leq r$.

Exercise. Prove that for every $\mathfrak{a} \in \mathcal{D}$, $\mathfrak{a} \geq 0$, and $k \geq 1$ integer,

$$\dim((k-1)\mathfrak{a}) \leq \dim(k\mathfrak{a}) \leq \dim((k-1)\mathfrak{a}) + \deg \mathfrak{a}.$$