Ramification

Definition

L/K finite field extension. A D.D with $K = \operatorname{Frac} A$.

Let B be the i.c of A in L. Assume that

B is a f.g. A-module.

Let $M \in \operatorname{Max} B$, $P = M \cap A$.

we proved that since $\dim A = 1$ and B/A is integral then $P \in \operatorname{Max} A$.

f.g. A-module

$$\begin{array}{ccc} M & B & L \\ \downarrow e_{M/P} & | & | \;\; \} \text{finite} \\ P = M \cap A & A & K \end{array}$$

The ideal M is ramified over P (or over A really) if

1) $e_{M/P} > 1$ or

2) B/M is not separable over A/P.

If the ideal M is not ramified over P it is said to be unramified.

Remark

A first guess for what should be the definition of ramification would probably be (1) above. (2) seems strange. It will turned out (already in the next unit) that (1) or (2) lends itself to a very clean description. Secondly, as we'll see, in our setting of curves over finite fields, separability of the residue field extensions won't be an issue.

Definition

With A, B, K, L as in the previous definition, a maximal ideal P of A <u>ramifies in B</u> if PB is contained in a maximal ideal M of B which is ramified over P.

When no max ideal of B is ramified over A, the extension B/A is called <u>unramified</u>.

Example

Let $f(x,y) = y^2 - x$, and consider $Z_f(\mathbb{C})$. Let $C_f = \mathbb{C}[x,y]/\langle y^2-x\rangle$

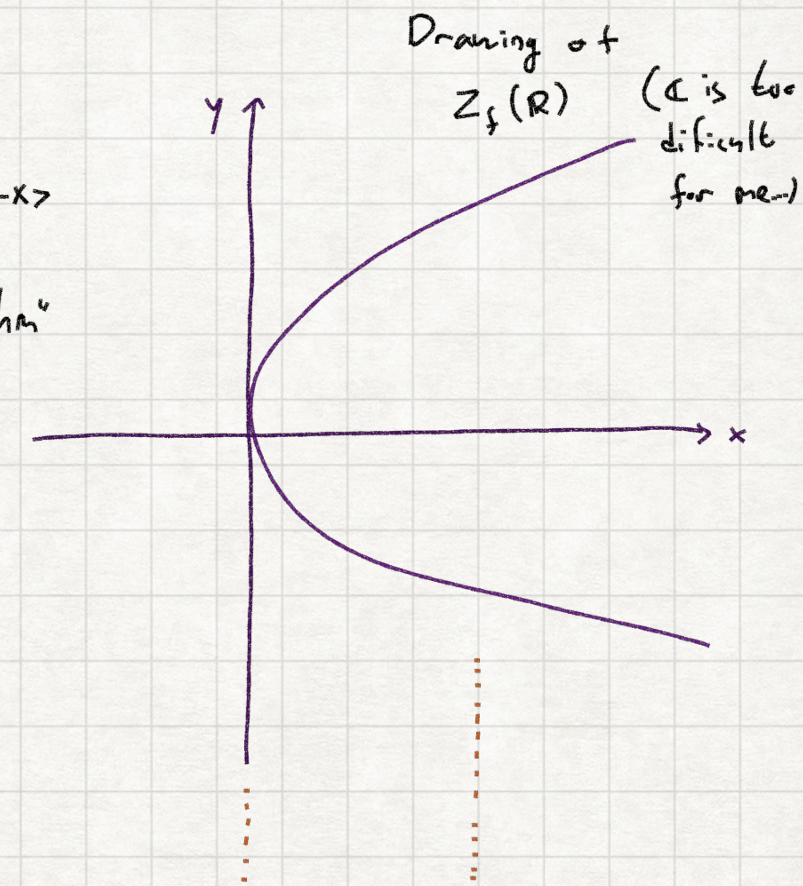
We'll factor xC_f and $(x-1)C_f$. Recall our "algorithm"

from the previous unit: To factor $(x-t)C_f$,

write $f(x,y) = y^2 - x$ as $(x-t)g(x,y) + \prod_{i=1}^{s}(y-b_i)^{e_i}$.

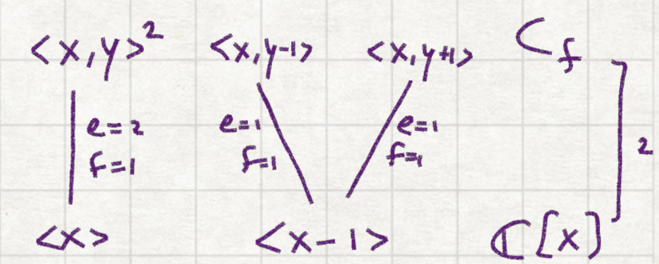
Then, $(x-a)C_f = \prod_{i=1}^{s} M_i^{e_i}$ where $M_i = (x-a)C_f + (y-b_i)C_f$

$$= \langle x-a,\ y-b_i\rangle$$



The ideal $\langle x\rangle$

$y^2 - x = (-1)x + y^2$ and so $\langle x\rangle = \langle x,y\rangle^2$.



It is easy to verify this directly:

$$\langle x,y\rangle^2 = \underbrace{\langle x^2\rangle}_{\subseteq\langle x\rangle} + \underbrace{\langle xy\rangle}_{\subseteq\langle x\rangle} + \underbrace{\langle y^2\rangle}_{=\langle x\rangle} = \langle x\rangle$$

Further, $\langle x,y\rangle$ is max in C_f since
$$C_f/\langle x,y\rangle = \left(\mathbb{C}[x,y]/\langle y^2-x\rangle\right)/\langle x,y\rangle$$
$$\cong \mathbb{C}[x]/\langle x\rangle \cong \mathbb{C}.$$

The ideal $\langle x-1\rangle$

$$y^2 - x = -(x-1) + y^2 - 1 = -(x-1) + (y+1)(y-1)$$

$$\Rightarrow \quad \langle x-1\rangle = \langle x, y+1\rangle\langle x, y-1\rangle.$$

In general, over $\mathbb{C}$

$$y^2 - x = -(x-t) + y^2 - t = -(x-t) + (y+\sqrt{t})(y-\sqrt{t}).$$

For any $t \neq 0$ we get two distinct maximal ideals: $\langle x, y-\sqrt{t}\rangle$, $\langle x, y+\sqrt{t}\rangle$

over $\langle x\rangle$.

By the fundamental equality (note that $C_f = \mathbb{C}[x,y]/\langle y^2-x\rangle \cong \mathbb{C}[x]$ is a PID $\Rightarrow$ D.D).

$\sum_{i=1}^{s} e_i f_i = 2$. Since for any $t \neq 0$ $e_1 = 2 \Rightarrow s = 1$, $f_1 = 1$. For $t = 0$ we showed that

$s = 1$, $e = 2$, $f = 1$.