Algebraic Geometric Codes

Recitation 14

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A Tower over $\mathbb{F}_4$

Consider the tower $\mathcal{T}_1 := F_0 = \mathbb{F}_4(x_0) \subseteq F_1 = \mathbb{F}_4(x_0, x_1) \subseteq \ldots$ over $\mathbb{F}_4$ defined by the equation

$$Y^3 = \frac{X^3}{X^2 + X + 1}.$$

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We will study this tower, in two ways.

Rational places

F_0 has 5 rational places: $\mathfrak{p}_\infty$, $\mathfrak{p}_0$, $\mathfrak{p}_1$, $\mathfrak{p}_\delta$, $\mathfrak{p}_{\delta+1}$. How do they split in F_1?.

- $\mathfrak{p}_\infty$: $v_\infty(\frac{x_0^3}{x_0^2+x_0+1}) = -1$. Thus $\tilde{v}_\infty(\frac{x_0^3}{x_0^2+x_0+1}) = -e(\mathfrak{P}/\mathfrak{p})$.

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- $\mathfrak{p}_0$: here, Kummers theorem will not help as the corresponding polynomial is $Y^3 - \mathfrak{p}_0(\frac{x_0^3}{x_0^2+x_0+1}) = Y^3 - 0$ which promises use one place. Fortunately, we can write $(\frac{x_1}{x_0})^3 = \frac{1}{x_0^2+x_0+1}$. Thus from Kummers theorem $\mathfrak{p}_0$ splits completely, into $\mathfrak{P}_0^1, \mathfrak{P}_0^2, \mathfrak{P}_0^3$ where $\mathfrak{P}_0^i(x_1) = 0$, and $\mathfrak{P}_0^i(\frac{x_1}{x_0}) \in \{1, \delta, \delta+1\}$.

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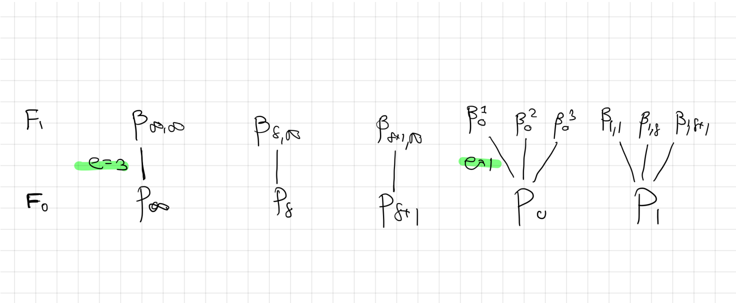
Finally, From Kummers theorem no place of degree ≥ 2 can be ramified (as it is not a zero or pole of $\frac{x_0^3}{x_0^2+x_0+1}$).

Illustration



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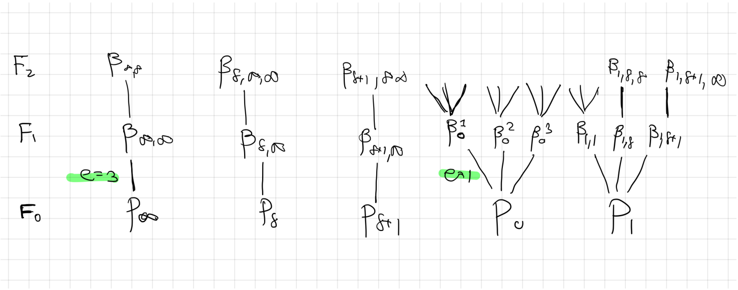

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Therefore

$$\lambda(\mathcal{T}_1) \geq \lim_{i \to \infty} \frac{3^i}{3^i - i - 1} = 1 = \sqrt{4} - 1$$

Simpler calculation

Recall,

Definition 1

Let $\mathcal{F}$ be a tower over $\mathbb{F}_q$. The set

$$\mathrm{Split}(\mathcal{F}) = \{\mathfrak{p} \in \mathbb{P}_1(F_0) \mid \mathfrak{p} \text{ splits completely in all extesnions } F_i/F_0\}$$

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Let F/L be an extension of E/K. A prime divisor $\mathfrak{p}$ of E/K is said to *ramify* in the extension F/L of E/K if $\exists \mathfrak{P}/\mathfrak{p}$ s.t. $e(\mathfrak{P}/\mathfrak{p}) > 1$.

Definition 2

Let $\mathcal{F}$ be a tower over $\mathbb{F}_q$. The set

$$\text{Ram}(\mathcal{F}) = \{\mathfrak{p} \in \mathbb{P}(F_0) \mid \mathfrak{p} \text{ is ramified in } F_i/F_0 \text{ for some } i \geq 1\}$$

is called the *ramification locus* of $\mathcal{F}$.

Simpler calculation

We saw in class that in Kummer extensions, if we denote $r = \sum_{p \in \mathrm{Ram}(\mathcal{F})} \deg \mathfrak{p}$ and $s = |\mathrm{Split}(\mathcal{F})|$ then

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In our example $\mathcal{T}_1$ we have that:
$\mathrm{Ram}(\mathcal{F}) = \{\mathfrak{p}_\infty, \mathfrak{p}_1, \mathfrak{p}_\delta, \mathfrak{p}_{\delta+1}\}$ and $\mathrm{Split}(\mathcal{F}) = \{\mathfrak{p}_0\}$, and so:

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Consider the variable transform $z_i = \frac{1}{x_i}$. We have here that $F_i = F_{i-1}(z_i)$ and the tower is defined by the equation

$$y^3 = (x+1)^3 - 1.$$

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Let ℓ be a prime power and $q = \ell^r$ for $r \geq 2$. Let $m = \frac{q-1}{\ell-1}$, note that m and ℓ are coprime. We will show that the sequence $\mathcal{T}_2 = (F_0, F_1, \ldots)$ that is recursively defined by

$$Y^m = (X+1)^m - 1.$$

is a tower over $\mathbb{F}_q$. So, we need to prove that

1. $F_i \neq F_{i+1}$;
2. F_{i+1}/F_i is separable
3. $\mathbb{F}_q$ is the constant field of F_i; and
4. $g(F_j) \geq 2$ for some j.

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To prove Items 1,2,3 using a claim from class we will find, for each $i \in \mathbb{N}$

$$\mathfrak{p}_i \in \mathbb{P}(F_i), \mathfrak{P}_i \in \mathbb{P}(F_{i+1}) \quad \text{s.t.} \quad \mathfrak{P}_i/\mathfrak{p}_i \quad \text{and} \quad e(\mathfrak{P}_i/\mathfrak{p}_i) = m.$$

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Let $\mathfrak{p}_0$ be the unique zero of x_0 in $F_0 = \mathbb{F}_q(x_0)$. Let $\mathfrak{P}_0/\mathfrak{p}_0$ in $\mathbb{P}(F_1)$. We have that

$$m \cdot v_{\mathfrak{P}_0}(x_1) = v_{\mathfrak{P}_0}(x_1^m) = e(\mathfrak{P}_0/\mathfrak{p}_0) \cdot v_{\mathfrak{p}_0}\left((x_0+1)^m - 1\right) = e(\mathfrak{P}_0/\mathfrak{p}_0).$$

Thus, using also the fundamental equality, $e(\mathfrak{P}_0/\mathfrak{p}_0) = m$ as desired.

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Lemma 3

Let $\mathcal{F} = (F_0, F_1, \ldots)$ be a recursive tower over $\mathbb{F}_q$ defined by the equation

$$f(Y) = h(X),$$

with a basic function field F. Define

$$\Lambda_0 := \{x(\mathfrak{p}) \mid \mathfrak{p} \in \mathbb{F}_q(x) \text{ is ramified in } \mathbb{F}_q(x,y)/\mathbb{F}_q(x)\} \subseteq \overline{\mathbb{F}_q} \cup \{\infty\}.$$

Suppose that $\Lambda \subseteq \overline{\mathbb{F}_q} \cup \{\infty\}$ satisfies:

1. *$\Lambda_0 \subseteq \Lambda$; and*
2. *$\forall \beta \in \Lambda$, any solution $\alpha \in \overline{\mathbb{F}_q} \cup \{\infty\}$ to the equation $f(\beta) = h(\alpha)$ in fact satisfies $\alpha \in \Lambda$.*

Then, the ramification locus $\mathrm{Ram}(\mathcal{F})$ is finite and

$$\mathrm{Ram}(\mathcal{F}) \subseteq \{\mathfrak{p} \in \mathbb{P}(\mathbb{F}_q(x_0)) \mid x_0(\mathfrak{p}) \in \Lambda\}.$$

Λ for $\mathcal{T}_2$

First we note that $(x+1)^m - 1$ splits into different prime factors as $gcd((x+1)^m - 1, m(x+1)^{m-1}) = 1$. Thus, this is a tame cyclic extension and

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$$Norm_{\mathbb{F}_\ell}(x) = \prod x^{\ell^i} = x^{\sum \ell^i} = x^m.$$

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Finally, as $Norm_{\mathbb{F}_\ell} : \mathbb{F}_q^\times \to \mathbb{F}_\ell^\times$ is a group homomorphism there are exactly $m = \frac{q-1}{\ell-1}$ solutions to the equation $Norm(x) = 1$ in $\mathbb{F}_q$, which are all the solutions.

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Let $\beta \in \mathbb{F}_q$, then and therefore, $\beta^m = Norm_{\mathbb{F}_\ell}(\beta) \in \mathbb{F}_\ell$ thus, as before, all the solutions to the equation $(\alpha+1)^m = \beta^m + 1$ are in $\mathbb{F}_q = \Lambda$.

Splitting locus for $\mathcal{T}_2$

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$\mathcal{T}_2$ is a asymptotically good tower

First, we must show that item 4 holds, i.e. for some i $g_i \geq 2$. Indeed,

$$2g_1 - 2 = [F_1 : F_0](2g_0 - 2) + \deg Diff(F_1/F_2)$$

plug in what we know,

$$2g_1 - 2 \geq -2 \cdot m + (m-1)(m) = (m-3)m \geq m \geq 2$$

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Now for $\Lambda(\mathcal{T}_2)$:

$$\lambda(\mathcal{T}_2) \geq \frac{2s}{r-2} \geq \frac{2 \cdot 1}{q-2} > 0$$

which implies that the code is asymptotically good.