Free Central Limit Theorem

Based on Nica-Speicher Chapter 8

The setup.

Let $(\mathcal{A},\varphi)$ be a $*$-ps.

Let $a_1, a_2, \ldots \in \mathcal{A}$ identically distributed self adjoint r.v.

which are tensor / free ind.

Assume they are centered: $\forall r \ \varphi(a_r)=0$, and denote

$$\sigma^2 := \varphi(a_r^2)$$

non-negative since

$\varphi(a_r^2) = \varphi(a_r^* a_r) \geq 0$

A central limit theorem asks about the behavior of

$$\frac{a_1 + \cdots + a_N}{\sqrt{N}}$$

This normalization is with hindsight

as $N \to \infty$.

The only meaningful convergence in FPT is in moments:

Def. Let $((\mathcal{A}_N, \varphi_N))_{N \in \mathbb{N}}$ & $(\mathcal{A}, \varphi)$ be ncps.

Let $a_N \in \mathcal{A}_N$, $N \in \mathbb{N}$, and $a \in \mathcal{A}$. We say that a_N converges in distribution towards a, and write $a_N \xrightarrow{\text{dist}} a$ if

$$\forall n \in \mathbb{N} \quad \lim_{N \to \infty} \varphi_N(a_N^n) = \varphi(a^n).$$

If a_N has analytic dist μ_N recall μ_N is compactly supported

and a has —"— μ,

all on $\mathbb{R}$ recall a_N, a are self adjoint

then the classical notion of "convergence in distribution"

means "weak-convergence" of μ_N towards μ:

$$\lim_{N\to\infty} \int f(t)\, d\mu_N(t) = \int f(t)\, d\mu(t) \qquad \forall \text{ bounded continuous } f.$$

Indeed, convergence of all moments $\Rightarrow$ convergence of all poly

$\Rightarrow$ convergence of bounded continuous functions.

Stone-Weierstrass

However, the normal dist does not have compact support. It is nonetheless "nice" enough:

Def. Let μ be a probability measure on $\mathbb{R}$ with moments

$$m_n = \int t^n \, d\mu(t)$$

We say that μ is <u>determined by its moments</u> if μ is the only dist on $\mathbb{R}$ with these moments.

* We can generalize our def of dist in the analytic sense to accomodate these dist.

Fact. The normal dist is det by its moments.

* Let μ & (μ_N) be prob measures on $\mathbb{R}$ s.t μ is det by its moments & (μ_N) have moments of every order &

$$\lim_{N\to\infty} \int t^n \, d\mu_N(t) = \int t^n \, d\mu(t) \quad \forall n$$

$\Rightarrow$ μ_N converges weakly to μ.

Cor. For the weak conv of classical rv to the normal dist it suffices to check conv of all moments.

Fix $n \geq 1$ & $N \geq 1$ integers.

$$\varphi((a_1 + \dots + a_N)^n) = \sum_{1 \leq r(1), \dots, r(n) \leq N} \varphi(a_{r(1)} \cdots a_{r(n)})$$

Since a_r-s have the same distribution, we have e.g.,

$$\varphi(a_1 a_2 a_2 a_3 a_1 a_2) = \varphi(a_9 a_1 a_1 a_5 a_9 a_1).$$

Generally For both tensor & free ind

$$\begin{array}{c} \varphi(a_{r(1)} \cdots a_{r(n)}) \\ \| \\ \varphi(a_{p(1)} \cdots a_{p(n)}) \end{array} \quad \Longleftarrow \quad \forall i,j \in [n] \quad r(i) = r(j) \iff p(i) = p(j)$$

That is, the value $\varphi(a_{r(1)} \cdots a_{r(n)})$ depends on $r(\cdot)$ only through the information on which of the indices are equal!

We'll encode this information by a partition

$$\pi = \{V_1, \dots, V_s\} \quad \text{of} \quad [n]$$

where

$$\forall i,j \in [n] \quad r(i) = r(j) \iff \overbrace{i,j \in V_m \text{ for some } m \in [s]}^{i \sim_\pi j}$$

In such case we write $(r(1), \dots, r(n)) \triangleq \pi$.

E.g. both $\underset{1}{a_1}\,\underset{2}{a_2}\,\underset{3}{a_2}\,\underset{4}{a_3}\,\underset{5}{a_1}\,\underset{6}{a_2}$ & $a_9\,a_1\,a_1\,a_5\,a_9\,a_1$

correspond to $\pi = \{\{1,5\}, \{2,3,6\}, \{4\}\}$

With this we can write

number of summands is independent of N!

N comes only through this

$$\varphi((a_1+\dots+a_N)^n) = \sum_{\pi \text{ partition of } [n]} \kappa_\pi \cdot A_\pi^N$$

$$A_\pi^N \triangleq \left| \{ (r(1),\dots,r(n)) \in [N]^n : (r(1),\dots,r(n)) \triangleq \pi \} \right|$$

The common value of $\varphi(a_{r(1)} \cdots a_{r(n)})$ for all $(r(1),\dots,r(n)) \triangleq \pi$

Observation. π containing a singleton do not contribute:

Indeed, if $\pi = \{V_1,\dots,V_s\}$ & $V_m = \{r\}$ then

Holds both for tensor & free ind

Skip

$$\kappa_\pi = \varphi(a_{r(1)} \cdots a_r \cdots a_{r(n)}) = \underbrace{\varphi(a_r)}_{=0}\,\varphi(a_{r(1)} \cdots \check{a}_r \cdots a_{r(n)})$$

as a_r is centered

In particular, we can restrict to $\pi = \{V_1, \dots, V_s\}$ s.t. $s \leq \frac{n}{2}$.

With this lets compute A_π^N:

$\pi = \{V_1, \dots, V_{|\pi|}\}$

$$A_\pi^N = N(N-1)\cdots(N-|\pi|+1) \qquad (\leq N^{N/2})$$

N options for an index appearing in the index set V_1

$N-1$ options for V_2

options for $V_{|\pi|}$

n remains fixed

$$\Longrightarrow \lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\dots+a_N}{\sqrt{N}}\right)^n\right) = \lim_{N\to\infty} \sum_\pi \frac{A_\pi^N}{N^{N/2}} K_\pi$$

$$= \sum_\pi K_\pi \cdot \lim_{N\to\infty} N^{|\pi|-\frac{N}{2}}.$$

$$\sum_{\pi} \kappa_{\pi} \cdot \underbrace{\lim_{N\to\infty} N^{|\pi| - \frac{N}{2}}}_{(*)}$$

aha pair partition

$$(*) = \begin{cases} 1 & \text{if } \pi \text{ is a pairing} \quad (\forall\, V_i \in \pi \quad |V_i| = 2) \\ 0 & \text{o.w.} \end{cases}$$

$$\Longrightarrow \lim_{N\to\infty} \varphi\left(\left(\frac{a_1 + \cdots + a_N}{\sqrt{N}}\right)^n\right) = \sum_{\substack{\pi \text{ pairing} \\ \text{of } [n]}} \kappa_\pi .$$

CLT for tensor independence

(aka classical CLT)

In the classical case $\forall$ pairing π $\quad \kappa_\pi = \sigma^n$

and

even n...

who is paired to some fixed remaining element

$$\#\text{pairings of } [n] = (n-1)(n-3)\cdots 1$$

who is paired to 1

Thus,

$$\lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\cdots+a_N}{\sqrt{N}}\right)^n\right) = (n-1)(n-3)\cdots 1\cdot\sigma^n$$

Exercise. Prove that

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{\mathbb{R}} t^n e^{-\frac{t^2}{2\sigma^2}}\,dt = \begin{cases}\sigma^n(n-1)(n-3)\cdots 1 & n \text{ even}\\ 0 & n \text{ odd}\end{cases}$$

This concludes the proof of the classical CLT. ∎

CLT for

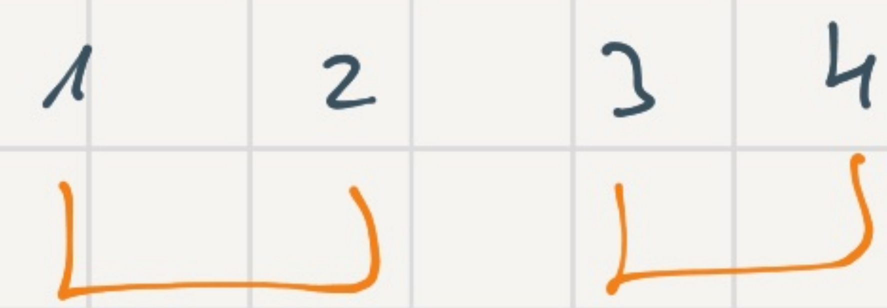
free independence

We turn to the free case. Recall

$$\lim_{N\to\infty} \varphi\left(\left(\frac{a_1+\cdots+a_N}{\sqrt{N}}\right)^n\right) = \sum_{\substack{\pi \text{ pairing} \\ \text{of } [n]}} K_\pi .$$

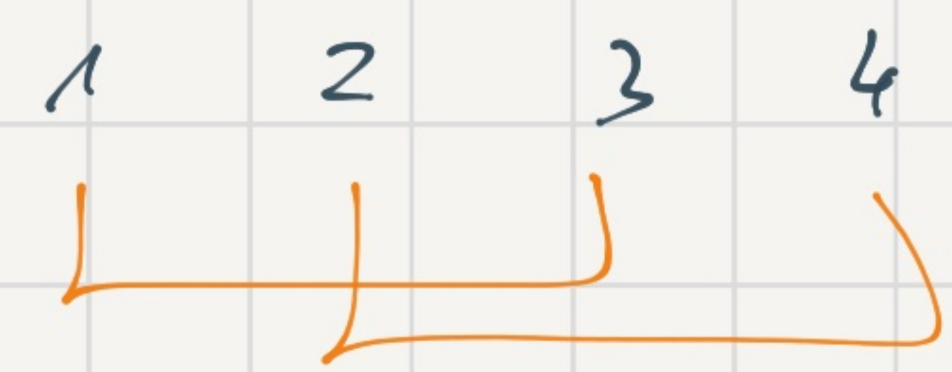
Lets look at some examples

$*$ $\pi = \{\{1,2\},\{3,4\}\}$



$$K_\pi = \varphi(a^2b^2) = \varphi(a^2)\varphi(b^2) = \sigma^4$$

$*$ $\pi = \{\{1,3\},\{2,4\}\}$



$$K_\pi = \varphi(abab) = 0$$

a & b are free and centered

$*$ $\pi = \{\{1,4\},\{2,3\}\}$

$$K_\pi = \varphi(ab^2a) = \varphi(b^2)\varphi(a^2) = \sigma^4$$

In general if π is such that $\{r, r+1\} \in \pi$ for some r then

a is free from $\{b, c\}$

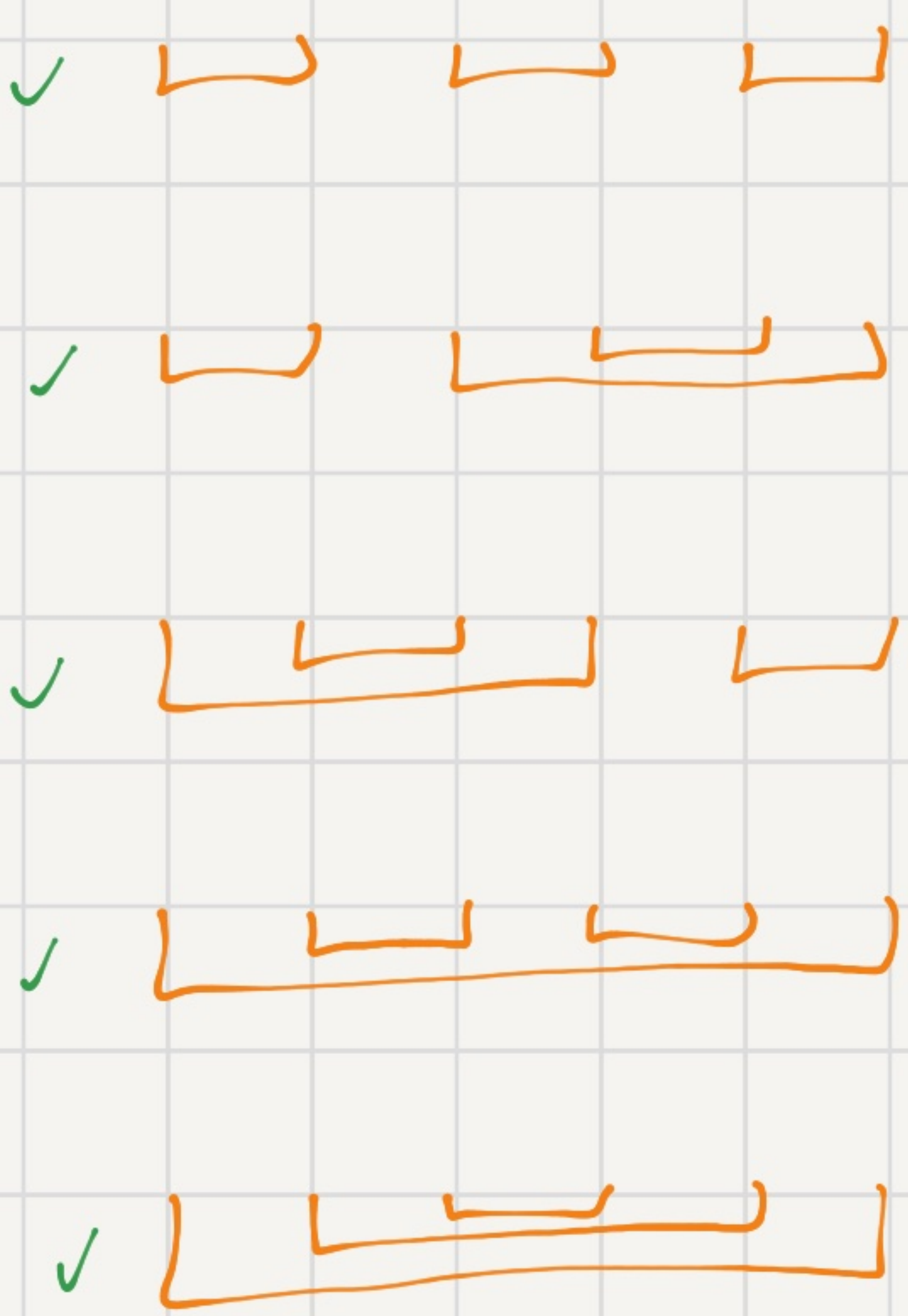
$$\varphi(b\, a_r a_{r+1} c) = \varphi(b a^2 c) = \varphi(a^2)\varphi(bc) = \sigma^2 \varphi(bc)$$

If the partition $\pi \setminus \{\{r, r+1\}\}$ also has a consecutive pairing (under the induced order on $[n]$, ignoring $r, r+1$) we can peal off another factor of σ^2 and repeat.

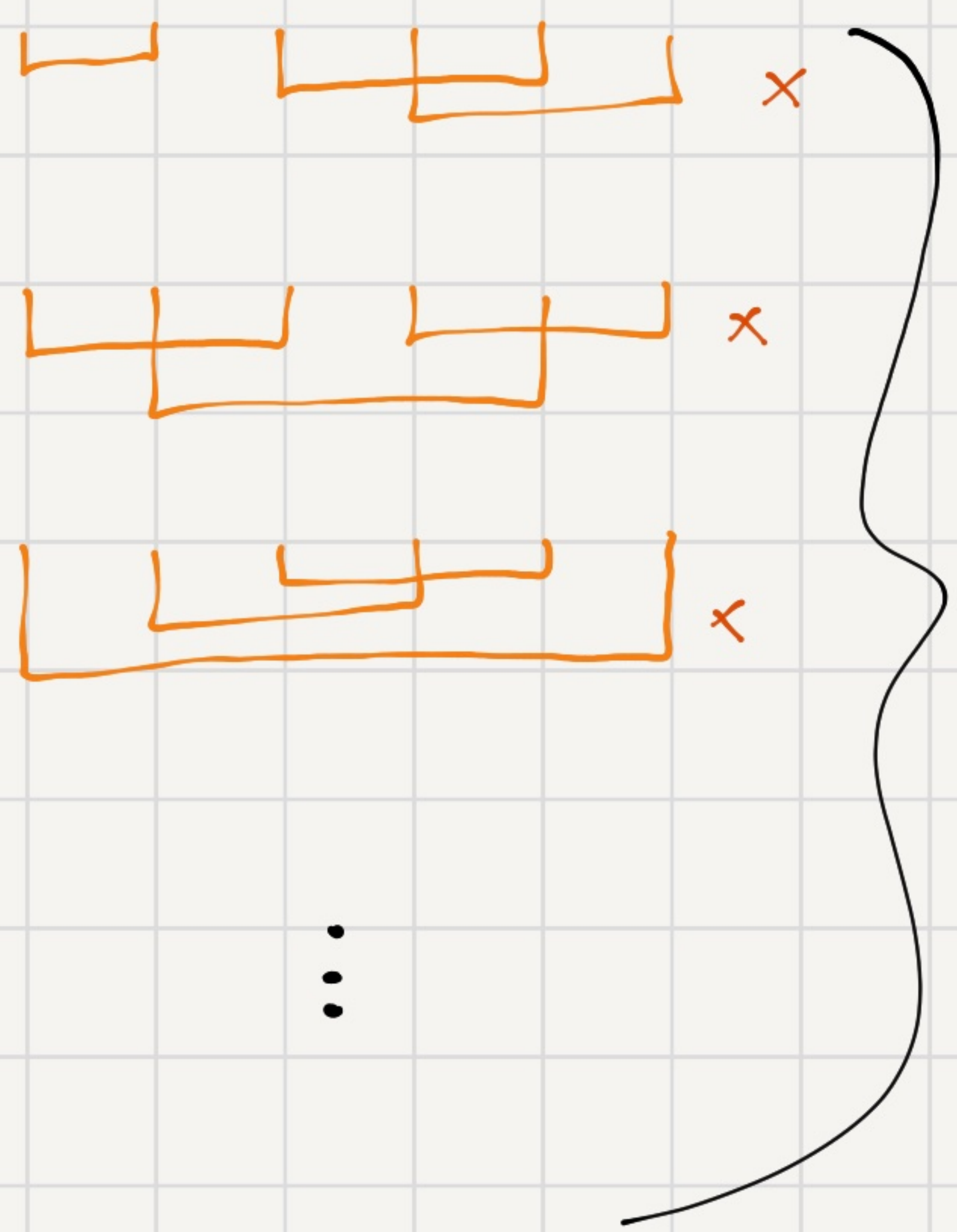
We will end up with $\kappa_\pi = \sigma^n$ unless no consecutive pair exists. This happens exactly when $r(1) \neq r(2) \neq r(3) \neq \cdots \neq r(n)$ in which case, by freeness, we get $\kappa_\pi = 0$.

For $n=4$ we had 2 such pairings:

$\underline{\underline{n=6}}$



5 good pairings

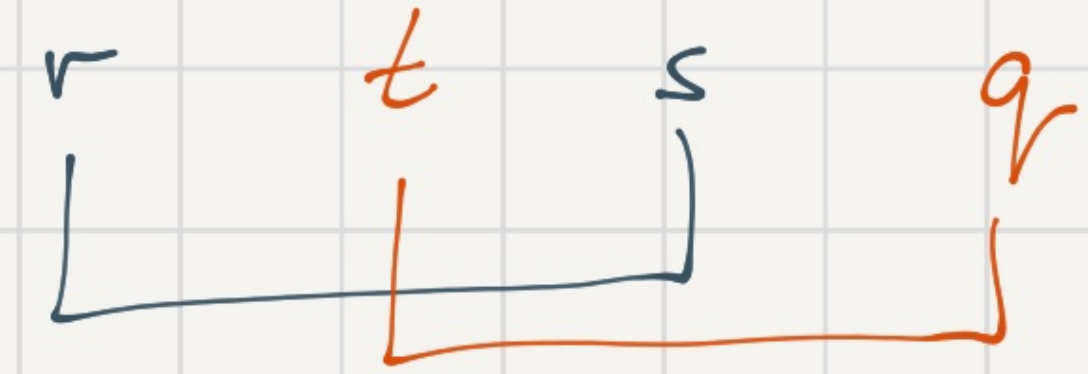


10 bad pairings

If π is s.t. $k_\pi = 0$ then at some point in the iterative process, we're looking at nonempty π' obtained from π with no consecutive pair. Take any pair $\{r<s\} \in \pi'$ s.t. there are no $r<r'<s'<s$ with $\{r',s'\} \in \pi'$ (why such $\{r,s\}$ exists?).

Recall π has no consecutive pair

Then $\exists t : r<t<s$ which is paired to some q. Thus $\{r,s\}$ & $\{t,q\}$ "cross". This holds also in π, of course.

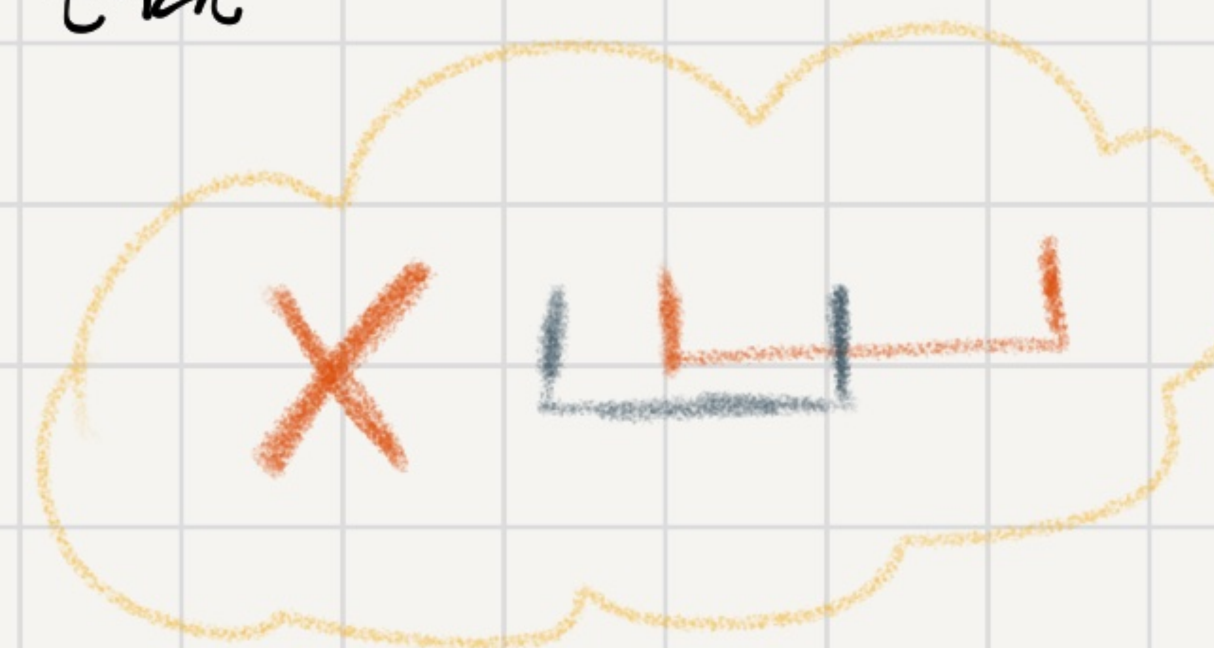


or

If $k_\pi \neq 0$ then there are no crossing pairs (verify).

Def. A pairing π of $[n]$ is called non-crossing if for every $\{p_1 < p_2\}, \{q_1 < q_2\} \in \pi$ it does not hold that

$$p_1 < q_1 < p_2 < q_2$$



The set of non-crossing pairings of $[n]$ is denoted by $NC_2(n)$.

Lemma. For n even, $n = 2k$, $|NC_2(n)| = C_k$.

pf. Denote $D_k = |NC_2(2k)|$. We know $D_1 = 1 = C_1$, so it is enough to show that D_n satisfies the same recurrence relation as C_n.

Let $\pi = \{V_1, \dots, V_k\}$ be a non-crossing pairing of $[2k]$. Assume WLOG that $1 \in V_1$. Then, $V_1 = \{1, m\}$. By non-crossing, $\forall\, 2 \leq j \leq k$, either $V_j \subseteq \{2, \dots, m-1\}$ or $V_j \subseteq \{m+1, \dots, n\}$. In particular, m is even, $m = 2\ell$. Thus, π restricted to $\{2, \dots, m-1\}$ is non-crossing & same for $\{m+1, \dots, 2k\}$. Thus,

$$D_{2k} = \sum_{\ell=1}^{k} D_{2(\ell-1)} \cdot D_{2(k-\ell)}$$

Recall that the Catalan numbers are the moments of the semicircular variable. Hence,

<u>Theorem (Free CLT).</u> Let $(\mathcal{A}, \varphi)$ be a $*$-ps. Let $a_1, \ldots \in \mathcal{A}$ be a sequence of freely independent identically distributed selfadjoint random variables s.t.

$$\forall r \quad \varphi(a_r) = 0 \quad \& \quad \varphi(a_r^2) = \sigma^2.$$

Then,

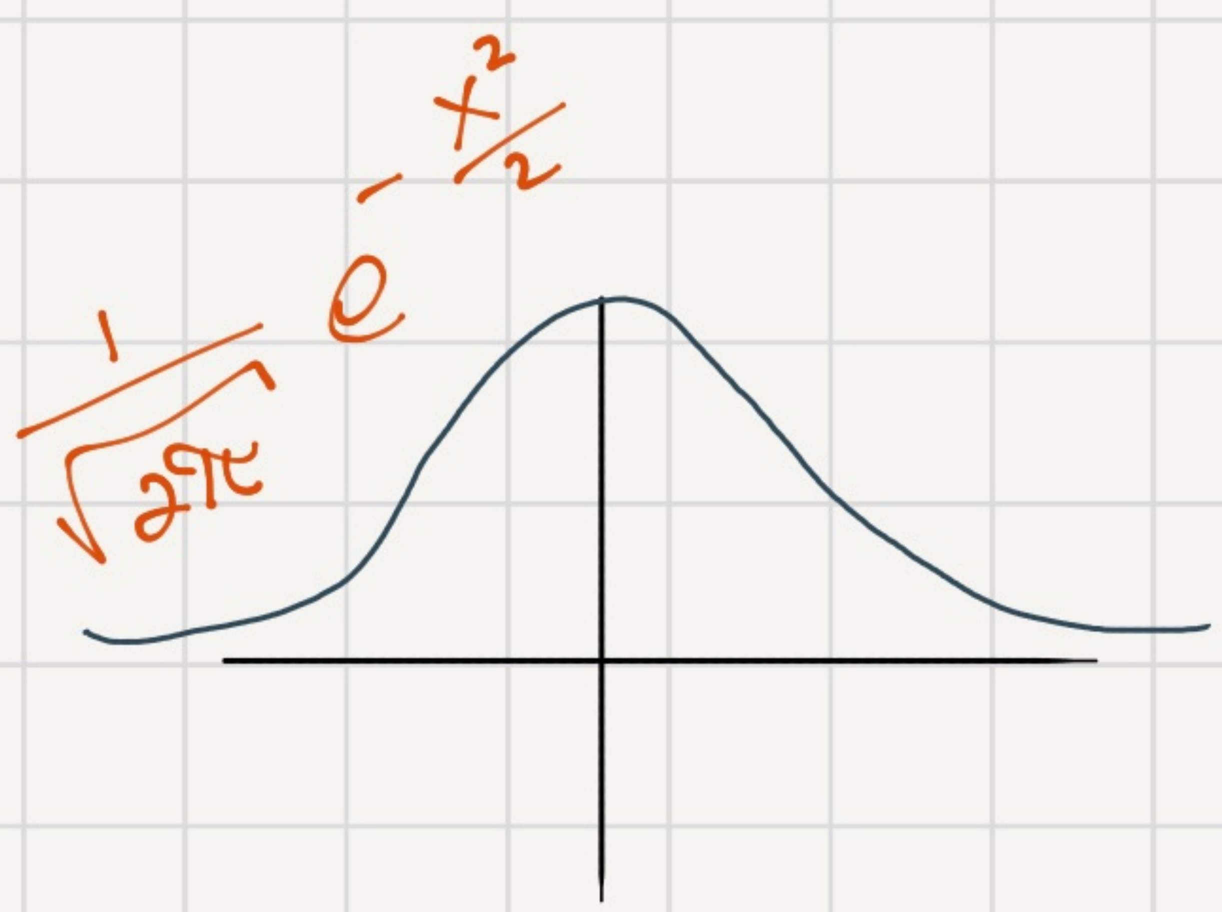
$$\frac{a_1 + \cdots a_N}{\sqrt{N}} \xrightarrow{\text{dist}} s$$

where s is a semicircular element of variance σ^2.

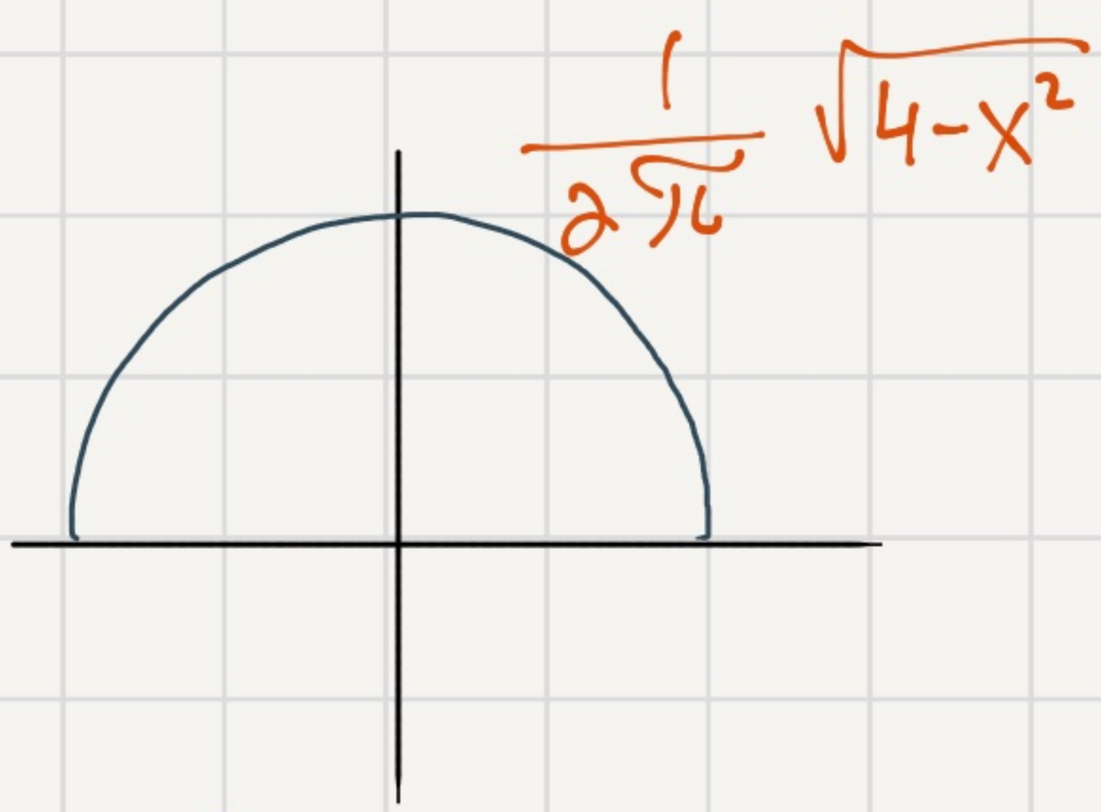
Remarks.

* The proofs for the CLTs suggest that on the combinatorial level classical probability is captured by pairings (and as we'll see later, partitions in general) whereas free ind is about <u>non-crossing</u> pairings / partitions.

* The semicircular distribution is the free analog of the normal distribution.



classical probability



free probability