Expander Graphs

Following Vadhan, Chapter 4

Gil Cohen

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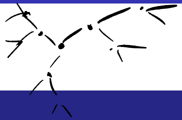
Outline

1. Three forms of expansion
2. Another view on spectral expanders
3. The expander mixing lemma
4. Hitting property of expander walks
5. Error reduction via expander random walks
6. The best spectral expanders - Ramanujan graphs

n^l

$(n+\omega)^t$

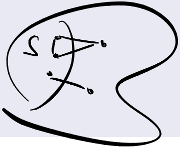
Vertex expansion



Definition

Let $G = (V, E)$ be an undirected graph. For $S \subseteq V$ we define the neighborhood of S by

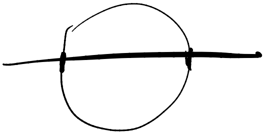
$$\Gamma(S) = \{u \in V \mid \exists v \in S \; uv \in E\}.$$



Definition

An undirected graph $G = (V, E)$ is a (k, a)-vertex expander if for every $S \subset V$ of size at most k it holds that $|\Gamma(S)| \geq a|S|$.

Vertex expansion



Theorem

For every $d \geq 3$ there is $\alpha > 0$ such that the following holds. For every integer $n \geq 1$ there exists a d-regular undirected graph on n vertices that is an $(\alpha n, d - 1.01)$-vertex expander.

$\alpha = \alpha(d, \text{"}0.01\text{"})$

Spectral expanders

$(1-\gamma)^t = \omega^t$

Definition

Let $G = (V, E)$ be an undirected graph. The spectral gap of G is defined by

$$\gamma(G) = \omega_1(G) - \omega(G) = 1 - \omega(G),$$

where, recall, $\omega(G) = \max(\omega_2(G), -\omega_n(G))$.

We say that G is a γ-spectral expander if $\gamma(G) \geq \gamma$.

Recall that the spectral gap is related to the rate of convergence of a random walk as

$$\omega(G) = \max_{\mathbf{p}} \frac{\|\mathbf{Wp} - \mathbf{u}\|}{\|\mathbf{p} - \mathbf{u}\|}$$

Spectral expanders

$\gamma = 1 - \omega$

Theorem

If G is a γ-spectral expander then it is an $(\frac{n}{2}, 1+\gamma)$-vertex expander.

To prove the theorem, we define

Definition

Let $\mathbf{p}$ be a distribution. The collision probability of $\mathbf{p}$ is the probability two independent samples from $\mathbf{p}$ are equal. Namely,

$$\mathrm{CP}(\mathbf{p}) = \sum_x \mathbf{p}(x)^2.$$

Spectral expanders

Lemma

For every probability distribution $\mathbf{p} \in [0,1]^n$,

1. $\mathrm{CP}(\mathbf{p}) = \|\mathbf{p}\|^2 = \|\mathbf{p} - \mathbf{u}\|^2 + \frac{1}{n}$.
2. $\mathrm{CP}(\mathbf{p}) \geq \frac{1}{|\operatorname{sup}(\mathbf{p})|}$.

$|supp(p)|$

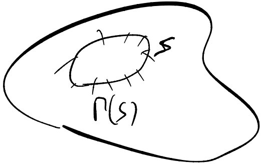
Spectral expanders

We now prove the theorem, restated below.

Theorem

If G is a γ-spectral expander then it is an $(\frac{n}{2}, 1+\gamma)$-vertex expander.

$$1-\gamma = \omega = \max_{p} \frac{\|Wp - \pi\|}{\|p-\pi\|}$$



$p =$ uniform dist on S

$$\|Wp - \pi\|^2 + \frac{1}{n} = CP(Wp) \geq \frac{1}{|Supp(Wp)|}$$

$$\|p-\pi\|^2 + \frac{1}{n} = cp(P) = \frac{1}{|S|}$$

Extra space for the proof

$$\|w_p-\pi\|^2 \geq \frac{1}{|\mathrm{supp}\, w_p|} - \frac{1}{n}$$

$$\|p-\pi\|^2 = \frac{1}{|S|} - \frac{1}{n}$$

$$\frac{\|w_p-\pi\|}{\|p-\pi\|} \leq \omega$$

$$\frac{1}{|\Gamma(S)|} - \frac{1}{n} \leq \|w_p-\pi\|^2 \leq \omega^2\left(\frac{1}{|S|}-\frac{1}{n}\right)$$

$$\leq \omega^2 \frac{1}{|S|}$$

$$\frac{1}{|\Gamma(S)|} \leq \frac{1}{n} + \frac{\omega^2}{|S|}$$

$$|\Gamma(S)| \geq \frac{|S|}{\omega^2} = \frac{|S|}{(1-r)^2} \geq (1+r)|S|$$

Edge expansion

Definition

An undirected graph $G = (V, E)$ is a (k, ε)-edge expander if for every $S \subseteq V$ of size $|S| \leq k$,

$$|\partial(S)| \geq \varepsilon d(S).$$

Recall that the conductance of S for a d-regular graphs is

$$\phi(S) = \frac{|\partial(S)|}{\min(d(S), d(V \setminus S))} = \frac{|\partial(S)|}{d \min(|S|, |V \setminus S|).}$$

Hence, for simplicity, focusing on $k = \frac{n}{2}$, in an $(\frac{n}{2}, \varepsilon)$-edge expander every set of size at most $\frac{n}{2}$ has conductance at least ε.

Edge expansion vs spectral expansion

$[-1,1]$

$W_{new} = \frac{I+W}{2}$

$[0,1]$

By Cheeger's inequality

$$\frac{\nu_2}{2} \leq \phi(G) \leq \sqrt{2\nu_2}.$$

Now,

$$1 - \gamma = \omega = \max(\omega_2, -\omega_n) \geq \omega_2 = 1 - \nu_2,$$

and so $\phi(G) \geq \frac{\nu_2}{2} \geq \frac{\gamma}{2}$.

For the other direction, we need to make sure $-\omega_n \leq \omega_2$. One way is to add sufficiently many self loops so to guarantee $\omega_n \geq 0$.

Spectral norm

Definition

Let $\mathbf{A}$ be a real matrix. The spectral norm of $\mathbf{A}$, denoted by $\|\mathbf{A}\|$, is given by

$$\|\mathbf{A}\| = \max_{0 \neq \mathbf{x} \in \mathbb{R}^n} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}.$$

Geometrically, $\|\mathbf{A}\|$ measures the largest "stretch" $\mathbf{A}$ can have.

Spectral norm

Lemma

The spectral norm of $\mathbf{A}$*, equals to the square root of the largest eigenvalue of* $\mathbf{A}^T\mathbf{A}$*. In particular, when* $\mathbf{A}$ *is symmetric,*

$$\|\mathbf{A}\| = \max\{|\alpha| : \alpha \in \mathrm{Spec}(\mathbf{A})\}.$$

$$\frac{\|Ax\|^2}{\|x\|^2} = \frac{x^T A^T A x}{x^T x}$$

Spectral norm

Lemma

We have the following properties of the spectral norm.

- *Subadditivity:* $\|\mathbf{A}+\mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$.
- *Submultiplactivity:* $\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$.
- $\|a\mathbf{A}\| \leq |a| \; \|\mathbf{A}\|$.

Another view on spectral expanders

Lemma

Let $G = (V, E)$ be an undirected regular graph. Then, G is a γ-spectral expander if and only if

$$\frac{1}{d}M_G = A_G = \mathbf{W}_G = \gamma \underset{=}{\mathbf{J}} + \overbrace{(1-\gamma)\mathbf{E}}^{\omega},$$

where $\mathbf{J}$ stands for the $n \times n$ all $\frac{1}{n}$ matrix, and $\|\mathbf{E}\| \leq 1$.

$$E = \frac{W - \gamma J}{1-\gamma}$$

לכיון

Extra space for the proof

Another view on spectral expanders

It is sometimes more convenient to decompose $\mathbf{W}_G$ as $\mathbf{W}_G = \underset{=}{\mathbf{J}} + \underset{=}{\mathbf{E}}$, where $\|\mathbf{E}\| \le \omega$.

$$\gamma J + \underbrace{(1-\gamma) E}_{E}$$

$$W = \sum_{i=1}^{n} \omega_i \varphi_i \varphi_i^T = J + \underbrace{\sum_{i=2}^{n} \omega_i \varphi_i \varphi_i^T}_{\|E\| \le \omega}$$

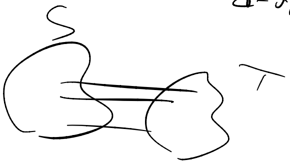
$$\omega_1 = 1$$

$$\varphi_1 = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix} = \sqrt{n}$$

$$\varphi_1 \varphi_1^T = J$$

Outline

d-reg



$$\underbrace{|S|d}_{\text{מס' ניסיונות}} \cdot \underbrace{\frac{|T|}{n}}_{\text{הסתברות להצלחה}}$$

The expander mixing lemma

Given two sets S, T of vertices, we denote

$$e(S,T) = \{uv \in E \mid u \in S, v \in T\}.$$

Lemma (The expander mixing lemma)

Let G be a d-regular $\gamma = 1 - \omega$ spectral expander on n vertices. Let $S, T \subseteq V$ be sets of density α, β respectively. Then,

$$\left| \frac{|e(S,T)|}{nd} - \alpha\beta \right| \leq \omega\sqrt{\alpha(1-\alpha)\beta(1-\beta)}.$$

$$\frac{e(S,T)}{nd} = \frac{|S| \cdot |T|}{n \cdot n} = \alpha\beta$$

Extra space for the proof

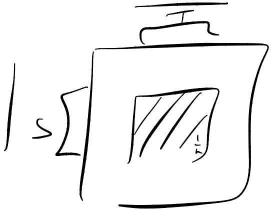
$$\mathbb{1}_S^T M \mathbb{1}_T = |e(S,T)|$$

$$= \sum_{uv \in E} M_{uv} \mathbb{1}_S(u) \mathbb{1}_T(v)$$

$$\mathbb{1}_S^T W \mathbb{1}_T = \frac{1}{d} |e(S,T)|.$$

$$\mathbb{1}_S^T J \mathbb{1}_T = n\alpha\beta = \frac{|S||T|}{n}$$

$$\mathbb{1}_S^T \tfrac{1}{\sqrt{n}} \tfrac{1}{\sqrt{n}}^T \mathbb{1}_T = \frac{|S|}{\sqrt{n}} \frac{|T|}{\sqrt{n}}$$



$$\sqrt{n} = \psi_1$$

Extra space for the proof

$$W = J + E \qquad \|E\| \le \omega$$

$$\frac{1}{n} 1_S^T W 1_T = \alpha\beta + \frac{1_S^T E 1_T}{n}$$

$$\frac{e(S,T)}{nd}$$

$$\left| \frac{e(S,T)}{nd} - \alpha\beta \right| \le \frac{1}{n} 1_S^T E 1_T$$

$$E = \sum_{i=2}^{n} \omega_i \varphi_i \varphi_i^T$$

$$\le \sum_{i=2}^{n} |\omega_i| |1_S^T \varphi_i \varphi_i^T 1_T|$$

$$\le \omega \sqrt{\sum_{i=2}^{n} (1_S^T \varphi_i)^2} \cdot \sqrt{\sum_{i=2}^{n} (1_T^T \varphi_i)^2}$$

$$\sqrt{\|1_T\|^2 - (1_T^T \varphi_1)^2}$$

$$\le \sqrt{|T| - \frac{|T|^2}{n}}$$

??

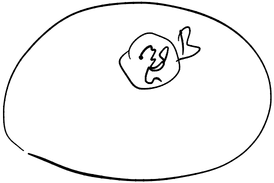
Hitting property of expander walks

Theorem

Let $G = (V, E)$ be a d-regular $\gamma = 1 - \omega$ spectral expander. Let $v_1, \ldots, v_t$ be a random walk in which v_1 is sampled uniformly at random from V. Then, for every $B \subseteq V$ having density μ,

$$\Pr[\{v_1, \ldots, v_t\} \subseteq B] \leq (\mu + \omega)^t.$$

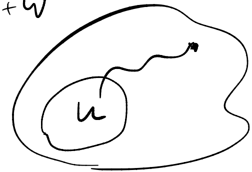
$(\mu + \sqrt{\omega})^t$



$\mu = \frac{|B|}{n}$

μ^t

$\mu^t + \omega$



Hitting property of expander walks

Claim

Let **P** *be the diagonal matrix indicating B. Then,*

$$u = \begin{pmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{pmatrix}$$

$$\Pr\left[\{v_1, \ldots, v_t\} \subseteq B\right] = \|(\mathbf{PW})^{t-1}\mathbf{Pu}\|_1$$
$$= \|(\mathbf{PWP})^{t-1}\mathbf{Pu}\|_1.$$

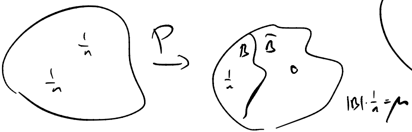
$$PW\underbrace{P\ \ P}W\underbrace{P\ \ P}W\underbrace{P\ \ P}u$$

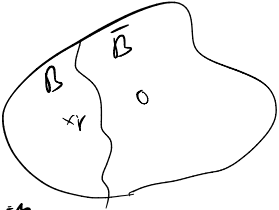
.

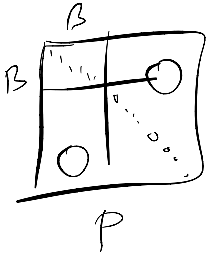
$t = 1$

$\|Pu\|_1$

u







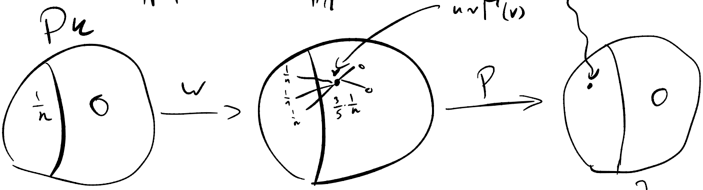
Extra space for the proof

$t=2$

$\|PWPu\|_1$

$\frac{1}{n} \Pr_{u\sim\Gamma(v)}[u \in B]$

$P = \begin{array}{cc} & B \\ B & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$



$$\|PWPu\|_1 = \sum_{v\in B} \frac{1}{n} \Pr_{u\sim\Gamma(v)}[u\in B]$$

$$\sum_{v\in V} \Pr[v\in B]\cdot \Pr_{u\sim\Gamma(v)}[u\in B]$$

Hitting property of expander walks

Claim

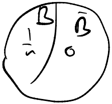
$$\cancel{\|\mathbf{PJP}\| \le \mu + \omega.}$$

$$\|(PWP)^{t-1}P_u\|_1 \le \sqrt{|B|}\cdot\|(PWP)^{t-1}P_u\|_2$$

$$\left(\|v\|_1 = \sum |v_i|\cdot 1 \le \sqrt{\sum v_i^2}\sqrt{\sum 1^2}\right)$$

$$\le \sqrt{|B|}\cdot\|PWP\|^{t-1}\cdot\underbrace{\|P_u\|}_{\frac{\sqrt{|B|}}{n}}$$

P_u



$$\|P_u\| = \sqrt{|B|\cdot\frac{1}{n^2}} = \frac{\sqrt{|B|}}{n}$$

$$W = \underbrace{J}_{(\frac{1}{n})} + \omega E \qquad \|E\| \le 1$$

$$\|PWP\| \le \|PJP\| + \omega\underbrace{\|PEP\|}_{\le 1}$$

Extra space for the proof

$\|PJP\|$

$\|PJPx\|$

$$\boxed{\begin{aligned}&\mu\cdot(\mu+\omega)^{t-1}\\ \le\ &(\mu+\omega)^{t}\end{aligned}}$$

$\alpha = \frac{1}{n}\sum_{v\in B} x_v$ ✓

$\|PJP\|$

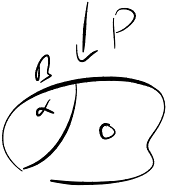
$\shortparallel$

$\|P_{\pi_B}^T P\|$

$\shortparallel$

$\|P_{\pi_B}\|^2$

$=$



$$\|PJPx\| = \sqrt{|B|\cdot\alpha^2} = \frac{\sqrt{|B|}}{n}\sqrt{\Big(\sum_{v\in B} x_v\Big)^2}$$

$$\le \frac{\sqrt{|B|}}{n}\cdot\sqrt{\sum_{v\in B} x_v^2}\cdot\sqrt{|B|} \le \mu\|x\|$$

Outline

Error reduction via expander random walks

Suppose we have a one-sided error randomized algorithm that uses r random bits and has constant error probability, say, $\frac{1}{2}$. Our goal is to reduce the error to ε with low cost in randomness.

Naively, we can run the algorithm $\log(1/\varepsilon)$ times, using fresh randomness each time, and return the AND (or OR) of the results. The randomness complexity is $r \cdot \log(1/\varepsilon)$.

Using expanders, we only need $r + O(\log(1/\varepsilon))$ random bits!

We remark that this savings can be obtained using pairwise independent distributions as well. Then, however, there is a $(1/\varepsilon)^{O(1)}$ blowup in time complexity.

A similar method works also for two-sided error, where the analysis is based on the expander Chernoff bound.

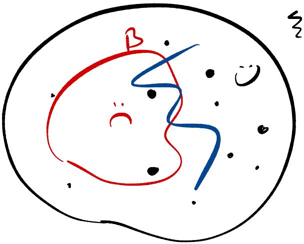
Extra space for the proof

$$\Pr\left[\left|\frac{|\{v_1,\dots,v_t\}\cap B|}{t}-\mu\right|\geq\varepsilon\right]$$

$$\leq e^{-c\cdot\varepsilon^2 t}$$

γ??

$\{0,1\}^r$



$\times\ \delta r$

$\omega=\frac{1}{4}$

$E\subset B$:

$$e^{-c\gamma\varepsilon^2 t}$$

$0\leq\gamma=1-\omega$

$$\Pr[\text{[illegible]}\ G\,] \leq \left(\tfrac{1}{2}+\omega\right)^t = \varepsilon$$

$t=O(\log\frac{1}{\varepsilon})$

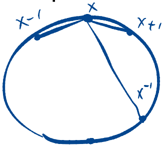
$$r + t\cdot\log d = r + O(\log\tfrac{1}{\varepsilon})$$

[illegible]

Ramanujan graphs

A natural question is how large can we make γ (equivalently, small ω) as a function of d? In the problem set, you will prove the Alon-Boppana bound

$$\omega \geq \frac{2\sqrt{d-1}}{d} - \varepsilon(n),$$



$\mathbb{F}_p$

3/16

where $\varepsilon(n) \to 0$. Remarkably, this is tight: there are graphs with

$$\omega \leq \frac{2\sqrt{d-1}}{d}.$$

$d \sim \frac{1}{\omega^2}$

Graphs meeting this bound are called Ramanujan graphs.

Interestingly, random d-regular graphs achieve, w.h.p, "only" $\omega \leq \frac{2\sqrt{d-1}}{d} + \varepsilon(n)$, with $\varepsilon(n) \to 0$.

Ramanujan graphs

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