Algebraic Geometric Codes

Recitation 04

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March 21, 2022

Valuation of the field of rational functions

Last week we categorized (up to equivalence) all the valuations of the field of rational functions $F(x)$. We proved that

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Theorem 1

Let $E/F(x)$ be an algebraic extension, and let $\varphi : F(x) \to L \cup \{\infty\}$ be a place, where L is algebraically closed. Then there is a place $\tilde{\varphi} : E \to L \cup \{\infty\}$ such that $\tilde{\varphi} \mid_{F(x)} = \varphi$.

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Note that the case that $\varphi = 0$ is trivial, so from now we assume $\varphi \neq 0$.

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Consider all the tuples

$$\{(R', \varphi') \mid R_0 \subseteq R' \text{ a subring of } E, \varphi' : R' \to L, \varphi' |_{R_0} = \varphi_0\}.$$

This set is not empty as it contains (R_0, φ_0). We can define an order on the tuples $(R'', \varphi'') \leq (R', \varphi')$ if $R'' \subseteq R'$ and $\varphi'|_{R''} = \varphi''$.

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If $\{(R_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is a chain, then $(\cup_{\alpha \in I} R_\alpha, \cup_{\alpha \in I} \varphi_\alpha)$ is an upper bound on the chain. Thus, by Zorn's lemma there is a maximal element in the set, denote it by (R, φ').

Step two – define the place

We want to prove that R is a valuation ring, and therefore we can extend $\tilde{\varphi} : E \to L \cup \{\infty\}$ by

$$\tilde{\varphi}(x) = \begin{cases} \varphi'(x) & x \in R \\ \infty & otherwise \end{cases}.$$

One can easily verify that if R is a valuation ring then $\tilde{\varphi}$ is a place, and $\tilde{\varphi}\mid_{F(x)} = \varphi$.

Step three – R is a valuation ring.

We will prove the following lemma.

Lemma 2

R has a unique maximal ideal $P = ker(\varphi')$.

Proof.

As $Im(\varphi')$ is contained in a field it follows that P is prime. As for the uniqueness and maximality, denote $R' = \{\frac{r}{s} \mid s \notin P\}$.
It holds that (a) $R \subseteq R'$. φ' can be extended to a morphism $\varphi'' : R' \to L$, and thus, as (R, φ') is maximal, it holds that $R = R'$. Therefore, if $s \in R \setminus P$ then $s^{-1} \in R$, thus $R \setminus P \subseteq R^{\times}$ ans the uniqueness follows. □

Denote that $K := \varphi'(R) \cong R/P$. $K \subseteq L$ is a subfield.

Step three – R is a valuation ring.

Lemma 3 (Chevalley's lemma)

φ' can be extended to a least one of the rings $R[z]$ or $R[z^{-1}]$.

Proof.

Extend φ' to a morphism $\psi : R[x] \to K[x]$ as follows: $\psi(\sum a_i x^i) = \sum \varphi'(a_i) x^i$. We want to use ψ, together with the substitution $\lambda : x \to z$ to extend φ' to $K[\alpha]$ for some $\alpha \in L$. Consider $I = \{f \in R[x] \mid f(z) = 0\}$ an ideal in $R[x]$. It holds that $\lambda(I) = 0$, thus we need to find α such that $g(\alpha) = 0$ for $g \in \psi(I)$. If we can find such α we are done. Note that as φ' is onto K, then *psi* is onto $K[x]$ and thus $\psi(I)$ is an ideal of $K[x]$. What are the possible cases:

□

Step three – R is a valuation ring.

Proof cont.

- $\psi(I) = 0$, then we can choose any $\alpha \in L$ and set $z \to \alpha$, to extend φ'.
- $\psi(I) = \langle 1f \rangle \subsetneq K[x]$. Then as L is algebraically closed we can choose α to be a root of f.
- Define $I' = \{f \in R[x] \mid f(z^{-1}) = 0\}$, then if one of the previous items holds for $\psi(I')$ we can extend $\varphi' : R[z^{-1}] \to L$.
- $\psi(I) = \psi(I') = K[x]$. Then we can not find such α. In this case there are f, g such that $f(z) = 0, \psi(f) = 1$ and $g(z^{-1}) = 0, \psi(g) = 1$. Assume both f, g have minimal degree. It holds that $\deg(f), \deg(g) \geq 1$. Assume w.l.o.g $n := \deg(f) \geq \deg(g) =: m \geq 1$. Consider the polynomial $g_0(z) = x^m g(x^{-1})$, $g_0(z) = z^m g(z^{-1}) = 0$, $\psi(g_0) = x^m$. Divide f in g_0 to obtain $f = g_0 q + r$ with $\deg(r) < m \leq n$.

Step three – R is a valuation ring.

Proof cont.

$0 = f(z) = g_0(z)q(z) + r(z) = 0 + r(z) \Rightarrow r(z) = 0$.
$1 = \psi(f) = \psi(g_0)\psi(q) + \psi(r)$ which implies $\psi(q) = 0$ and therefore $\psi(r) = 1$, and this is a contradiction to the minimally of f.

Thus, as (R, φ') are maximal, it follows that for every $z \in E^\times$ either $R[z] = R$ or $R[z^{-1}] = R$ and thus R is a valuation ring and the theorem holds.

Valuations

Corollary 4

Let $E/F(x)$ be an algebraic extension, and let $v : F(x) \to Z \cup \{\infty\}$ be a valuation. Then there is a place $\tilde{v} : E \to Z \cup \{\infty\}$ such that $\tilde{v}\,|_{F(x)}$ is equivalent to v.

Proof.

Let L be a field and let $\varphi : F(x) \to L \cup \{\infty\} \subseteq \overline{L} \cup \{\infty\}$ be the place corresponding to v. Let $R = \mathcal{Q}_v$ be v's valuation ring. Thus we can use Theorem 1 to construct $\tilde{\varphi} : E \to \overline{L} \cup \{\infty\}$. $\tilde{\varphi}$ is a place and so there is a corresponding valuation $\tilde{v}$. It is easy to verify that $\tilde{v}\,|_{F(x)}$ is a valuation and is indeed equivalent to v. □

Valuations

Corollary 5

Let E/F be a field extension and let $x \in E$ of transcendental degree 1. Let $x \in E$ be a transcendental element. Then there is valuations v, v' of E over F with $v(x) > 0, v'(x) < 0$.

Proof.

Follows by applying Corollary 4 with v_x, v_∞ of $F(x)$. $\square$