Free Independence

Based on Chapter 5 of

Nica-Speicher

but first...

Tensor (aka classical)

Independence

Def. Let $(\mathcal{A},\varphi)$ be a ncps

(1) Unital subalgebras $(\mathcal{A}_i)_{i\in I}$ are called **tensor independent** if they commute:

$$\forall i\neq j \quad \forall a\in\mathcal{A}_i, b\in\mathcal{A}_j \qquad ab=ba.$$

& if φ factorizes as follows:

$$\varphi\Big(\prod_{j\in J} a_j\Big)=\prod_{j\in J}\varphi(a_j) \qquad \begin{array}{l}\forall J\subseteq I \text{ finite } \& \\ \forall a_j\in\mathcal{A}_j \ (j\in J).\end{array}$$

(2) Tensor (or classical) independence of r.v is defined by tensor independence of the generated unital algebras.

So, a & b are tensor ind means that a, b commute and mixed moments factorize:

$$\varphi(a^n b^m) = \varphi(a^n)\varphi(b^m) \quad \forall n, m \geq 0$$

Remark. Tensor ind is an assumption (on the rv) that essentially gives a way of determining mixed moments by the individual (or marginal) moments. "Freeness" will be the second way of doing that.

we'll formalize that soon

finally...

Free

Independence

Def. Let $(\mathcal{A}, \varphi)$ be a ncps. Let I be an arbitrary index set.

(1) Let, for each $i \in I$, $\mathcal{A}_i \subset \mathcal{A}$ be a unital subalgebra.

The subalgebras $(\mathcal{A}_i)_{i \in I}$ are called <u>freely independent</u>

if

$$\varphi(a_1 \cdots a_k) = 0 \qquad (k \geq 1)$$

whenever:

* $a_j \in \mathcal{A}_{i(j)}$ $\quad j = 1, \dots, k$
* $\varphi(a_j) = 0$
* $i(1) \neq i(2) \neq i(3) \neq \cdots \neq i(k)$

Note it is still OK to have $i(1) = i(3)$

(2) Let $X_i \subset \mathcal{A}$ $(i \in I)$ be subsets of $\mathcal{A}$. Then, $(X_i)_{i\in I}$ are called freely independent if $(\mathcal{A}_i)_{i\in I}$ are freely ind, where $\mathcal{A}_i = \langle X_i, 1\rangle$

(3) In particular, if $\langle 1, a_i\rangle$, $a_i \in \mathcal{A}$ $i \in I$, are freely ind then $(a_i)_{i\in I}$ are called freely independent random variables.

(4) If $\mathcal{A}$ is a $*$-ps & the unital $*$-algebras $\mathcal{A}_i = \langle 1, a_i, a_i^*\rangle$ are freely ind, we say that $(a_i)_{i\in I}$ are $*$-freely independent.

As a matter of notation - we'll use "free" for "freely ind".

Lemma. Let $(\mathcal{A},\varphi)$ be a ncps, and let the unital subalgebra $(\mathcal{A}_i)_{i\in I}$ be free. Denote by $\mathcal{B} = \langle \mathcal{A}_i \mid i \in I \rangle$.
Then, $\varphi|_{\mathcal{B}}$ is uniquely determined by $(\varphi|_{\mathcal{A}_i})_{i\in I}$.

pf. By induction on k in the product $a_1 \cdots a_k$ $(a_j \in \mathcal{A}_{i(j)})$

Base case, $k=1$ is trivial.

Assume $k>1$. We may assume $i(1) \neq i(2) \neq \cdots$.

Let $a_j^\circ := a_j - \varphi(a_j) \in \mathcal{A}_{i(j)}$. Then,

$$\varphi(a_1 \cdots a_k) = \varphi\big((a_1^\circ + \varphi(a_1)) \cdots (a_k^\circ + \varphi(a_k))\big)$$

$$= \underbrace{\varphi(a_1^\circ \cdots a_k^\circ)}_{=0 \text{ (freeness)}} + \text{rest}$$

where

$$\text{rest} = \sum \underbrace{\varphi(\mathring{a}_{p(1)} \cdots \mathring{a}_{p(s)})}_{*} \varphi(a_{q(1)}) \cdots \varphi(a_{q(k-s)})$$

where we run over all disjoint decomposition

$$\{p(1) < \cdots < p(s)\} \,\dot{\cup}\, \{q(1) < \cdots < q(k-s)\} \qquad s < k.$$

As $s < k$, we can apply induction to $(*)$s.

<u>Notation.</u> The operation $a \longmapsto \mathring{a} = a - \varphi(a)$ is called <u>centering</u>.

Examples. For a, b free

$$* \quad 0 = \varphi\big((a-\varphi(a))(b-\varphi(b))\big)$$

$$= \varphi(ab) \underbrace{- \varphi(a\varphi(b)) - \varphi(\varphi(a)b) + \varphi(\varphi(a)\varphi(b))}_{\varphi(a)\varphi(b)}$$

$$\Longrightarrow \boxed{\varphi(ab) = \varphi(a)\varphi(b)}$$

Same as tensor ind

{a_1, a_2} & {b} are free

$$* \quad 0 = \varphi\big(\overset{\mathrm{I}}{(a_1-\varphi(a_1))}\overset{\mathrm{II}}{(b-\varphi(b))}\overset{\mathrm{IV}}{(a_2-\varphi(a_2))}\big)$$

$$= \varphi(a_1 b a_2) - \varphi(a_1 a_2)\varphi(b)$$

$$\Longrightarrow \boxed{\varphi(a_1 b a_2) = \varphi(a_1 a_2)\varphi(b)}$$

Same as tensor ind

I	II	III	
a_1	b	a_2	$\varphi(a_1 b a_2)$
a_1	b	$\varphi(a_2)$	$-\varphi(a_1)\varphi(b)\varphi(a_2)$
a_1	$\varphi(b)$	a_2	$-\varphi(b)\varphi(a_1 a_2)$
a_1	$\varphi(b)$	$\varphi(a_2)$	$\varphi(a_1)\varphi(b)\varphi(a_2)$
$\varphi(a_1)$	b	a_2	$-\varphi(a_1)\varphi(b)\varphi(a_2)$
$\varphi(a_1)$	b	$\varphi(a_2)$	$\varphi(a_1)\varphi(b)\varphi(a_2)$
$\varphi(a_1)$	$\varphi(b)$	a_2	$\varphi(a_1)\varphi(b)\varphi(a_2)$
$\varphi(a_1)$	$\varphi(b)$	$\varphi(a_2)$	$-\varphi(a_1)\varphi(b)\varphi(a_2)$

<u>Example</u> $\{a_1, a_2\}$ & $\{b_1, b_2\}$ are free.

$$\varphi(a_1 b_1 a_2 b_2) = \varphi(a_1 a_2)\varphi(b_1)\varphi(b_2) +$$

verify

$$\varphi(a_1)\varphi(a_2)\varphi(b_1 b_2) -$$

$$\varphi(a_1)\varphi(b_1)\varphi(a_2)\varphi(b_2).$$

In the tensor case, $\varphi(a_1 b_1 a_2 b_2) = \varphi(a_1 a_2)\varphi(b_1 b_2)$.

When are commuting r.v-s free?

When can commuting rv be free?

Assume a, b are commutative & free. Then,

$$\begin{aligned} \varphi(a^2 b^2) &= \varphi(a^2)\varphi(b^2) \\ \| \\ \varphi(abab) &= \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2 \end{aligned}$$

$$\Rightarrow \quad 0 = \underbrace{\left(\varphi(a^2) - \varphi(a)^2\right)}_{\varphi((a-\varphi(a))^2) = Var(a)} \left(\varphi(b^2) - \varphi(b)^2\right)$$

$\Rightarrow$ One of a, b has 0 variance.

If a (or b) is self-adjoint then $\operatorname{Var}(a)=0 \implies a$ is constant (a.e.).

$a=a^*$

$\varphi(a^*a)=0 \implies a=0$

More abstractly, if $\mathcal{A}$ is a $*$-ps & φ is faithful then $a \in \mathbb{C}$ (namely, a is constant). Indeed, write $a=\varphi(a)+b$.

$$a^2=\varphi(a)^2+2\varphi(a)b+b^2 \implies \varphi(a^2)=\varphi(a)^2+\varphi(b^2)$$

$\varphi(a^2)=\varphi(a)^2$ per our assumption

$$\implies \varphi(b^2)=0$$

$a=a^* \Downarrow b=b^*$

$$\implies \varphi(b^*b)=0$$

φ faithful

$$\implies b=0$$

Constants are
free from everything

Lemma. Let $(\mathcal{A},\varphi)$ be a ncps, and $\mathcal{B}\subseteq\mathcal{A}$ a unital subalgebra.

Then, the subalgebras $\mathbb{C}1$ & $\mathcal{B}$ are freely ind.

pf. Let $a_1,\dots,a_k$ as in the def of free ind. The case $k=1$ is trivial. If $k>1$ then at least one $a_i\in\mathbb{C}$ but then $\varphi(a_i)=0 \implies a_i=0$

$$0=\varphi(a_i)=a_i\cdot\varphi(1)=a_i\cdot 1$$

Hence, $a_1\cdots a_k=0 \implies \varphi(a_1\cdots a_k)=0$.

$\blacksquare$

Commutativity &
associativity of
freeness

__Remark__ (see problem set).

Freeness is commutative & associative. E.g.,

$$\mathcal{A}_1, \mathcal{A}_2 \text{ free} \iff \mathcal{A}_2, \mathcal{A}_1 \text{ free}$$

&

$$\begin{tikzcd}
\begin{matrix} X_1, X_2 \cup X_3 \text{ free} \\ \& \\ X_2, X_3 \text{ free} \end{matrix} \arrow[rr, Leftrightarrow] \arrow[dr, Leftrightarrow] & & \begin{matrix} X_1 \cup X_2, X_3 \text{ free} \\ \& \\ X_1, X_2 \text{ free} \end{matrix} \arrow[dl, Leftrightarrow] \\
& \begin{matrix} X_1, X_2, X_3 \\ \text{free} \end{matrix} &
\end{tikzcd}$$

Other probability

theories?

Are there other universal product measures?

Independence of subalgebras should give us a prescription for calculating a linear functional on the generated algebra.

So given $(\mathcal{A}_1, \varphi_1)$ & $(\mathcal{A}_2, \varphi_2)$ $\rightsquigarrow$ $(\langle \mathcal{A}_1, \mathcal{A}_2 \rangle, \varphi)$

Properties of the prescription:

* universal: independent of the subalgebras.
* $\varphi(a_1 b_1 a_2 b_2 \dots)$ involve products of moments s.t. in each product all a_i-s & all b_i-s appear exactly once and in the original order.

"Natural" construction in the language of Category Theory

$n=1$: $\quad \varphi(a_1 b_1) = \varepsilon \cdot \varphi_1(a_1)\varphi_2(b_1)$

universal means $\alpha, \dots, \varepsilon$ are ind of $\varphi_1, \varphi_2 \dots$

$n=2$: $\quad \varphi(a_1 b_1 a_2 b_2) = \alpha \varphi_1(a_1 a_2)\varphi_2(b_1 b_2) +$

$$\beta \varphi_1(a_1)\varphi_1(a_2)\varphi_2(b_1 b_2) +$$

$$\gamma \varphi_1(a_1 a_2)\varphi_2(b_1)\varphi_2(b_2) +$$

$$\delta \varphi_1(a_1)\varphi_1(a_2)\varphi_2(b_1)\varphi_2(b_2)$$

Constraints

* consistent wrt substitution
* associativity

$$1 = \varphi(1 \cdot 1) = \varepsilon \varphi_1(1)\varphi_2(1) = \varepsilon \implies \boxed{\varepsilon = 1}$$

$$\implies \varphi(ab) = \varphi_1(a)\varphi_2(b)$$

In particular, $\varphi|_A = \varphi_1$ $\quad \varphi|_B = \varphi_2$.

Now set $a_1 = a_2 = 1$ in $n=2$:

$$\varphi(1 \cdot b_1 \cdot 1 \cdot b_2) = \alpha \varphi_2(b_1 b_2) + \beta \varphi_2(b_1 b_2) +$$

$$\gamma \varphi_2(b_1)\varphi_2(b_2) + \delta \varphi_2(b_1)\varphi_2(b_2)$$

$$\implies \begin{cases} \alpha + \beta = 1 \\ \gamma + \delta = 0 \end{cases}$$

Similarly, putting $b_1 = b_2 = 1$,

$$\begin{cases} \alpha + \gamma = 1 \\ \beta + \delta = 0 \end{cases}$$

With this, setting $b_2 = 1$ we get

$$\begin{aligned} \varphi(a_1 b_1 a_2) &= (\alpha+\gamma)\varphi_1(a_1 a_2)\varphi_2(b_1) + \\ &\quad (\beta+\delta)\varphi_1(a_1)\varphi_1(a_2)\varphi_2(b_1) \\ &= \varphi_1(a_1 a_2)\varphi_2(b_1). \end{aligned}$$

We turn to exploit associativity:

Now lets consider three ps $(\mathcal{A}_i, \varphi_i)$ $i=1,2,3$ with elements

$a_1, a_2 \in \mathcal{A}_1$ $\quad b_1, b_2 \in \mathcal{A}_2$, $\quad c_1, c_2 \in \mathcal{A}_3$.

What is the ceff of $\varphi(a_1 a_2)\varphi(b_1 b_2)\varphi(c_1 c_2)$ in $\varphi(a_1 c_1 b_1 c_2 a_2 b_2)$?

$$\varphi(a_1(c_1 b_1 c_2) a_2 b_2) = \alpha \varphi(a_1 a_2) \underbrace{\varphi(c_1 b_1 c_2 b_2)}_{\alpha \varphi(b_1 b_2)\varphi(c_1 c_2) + \cdots} + \cdots \qquad \text{So } \boxed{\alpha^2}$$

$$\varphi((a_1 c_1) b_1 (c_2 a_2) b_2) = \alpha \underbrace{\varphi(a_1 c_1 c_2 a_2)}_{\varphi(a_1 a_2)\varphi(c_1 c_2) + \cdots} \varphi(b_1 b_2) + \cdots \qquad \text{So } \boxed{\alpha}$$

$$\Longrightarrow \alpha^2 = \alpha \qquad \Longrightarrow \alpha \in \{0, 1\}.$$

$\underline{\underline{\alpha = 0}}$ $\underline{\underline{\alpha = 1}}$

$\beta = 1$ $\beta = 0$

$\gamma = 1$ $\gamma = 0$

$\delta = -1$ $\delta = 0$

Free independence Tensor independence

(same phenomena in higher degree monomials)

$$\varphi(a_1 b_1 a_2 b_2) = \alpha \varphi_1(a_1 a_2)\varphi_2(b_1 b_2) + \beta \varphi_1(a_1)\varphi_1(a_2)\varphi_2(b_1 b_2) + \gamma \varphi_1(a_1 a_2)\varphi_2(b_1)\varphi_2(b_2) + \delta \varphi_1(a_1)\varphi_1(a_2)\varphi_2(b_1)\varphi_2(b_2)$$

$$\begin{cases} \alpha + \beta = 1 \\ \gamma + \delta = 0 \end{cases} \qquad \begin{cases} \alpha + \gamma = 1 \\ \beta + \delta = 0 \end{cases}$$