# Non-commutative probability spaces

Based on Nica-Speicher, Chapter 1

**Def.** A $\mathbb{C}$-algebra $\mathcal{A}$ is a vector space over $\mathbb{C}$ equipped with bilinear multiplication: $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ s.t.

not completely standard

* left & right distributivity: $(a+b)\cdot c = a\cdot c + b\cdot c$, $c\cdot(a+b) = c\cdot a + c\cdot b$ $\quad a,b,c \in \mathcal{A}$
* Compatibility with scalars: $(\alpha a)\cdot(\beta b) = (\alpha\beta)(a\cdot b)$ $\quad \alpha,\beta \in \mathbb{C}$
* Associative: $(a\cdot b)\cdot c = a\cdot(b\cdot c)$

$\mathcal{A}$ is unital if $\exists\, 1_{\mathcal{A}}$ – identity under multiplication

**Def.** A linear functional is a function $\varphi : \mathcal{A} \to \mathbb{C}$ s.t.

$$\varphi(\alpha a + \beta b) = \alpha\varphi(a) + \beta\varphi(b) \qquad \forall\, a,b \in \mathcal{A},\ \alpha,\beta \in \mathbb{C}$$

Assuming $\mathcal{A}$ is unital, $\varphi$ is unital if $\varphi(1_{\mathcal{A}}) = 1_{\mathbb{C}}$

**Def.** A non-commutative probability space (ncps) is

$$(\mathcal{A}, \varphi)$$

unital $\mathbb{C}$-algebra

unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$

The elements $a \in \mathcal{A}$ are called (nc) random variables.

Think of $\varphi(a)$ as $\mathbb{E}[a]$.

**Def.** $(\mathcal{A}, \varphi)$ is tracial if

$$\varphi(ab) = \varphi(ba) \qquad \forall a, b \in \mathcal{A}$$

So $\varphi(abc) = \varphi(cab)$
$= \varphi(bca)$
but not necessarily
$= \varphi(bac)$

**Def.** $\mathcal{A}$ is called a *$*$-algebra* if it is endowed with an antilinear $*$-operation

$$*: \mathcal{A} \to \mathcal{A}$$
$$a \mapsto a^*$$

$(ab)^* = b^* a^*$ $\forall a, b$

such that $(a^*)^* = a$ $\forall a$

**Def.** Let $(\mathcal{A}, \varphi)$ be a ncps. Assume $\mathcal{A}$ is a $*$-algebra s.t.

$$\forall a \in \mathcal{A} \qquad \varphi(a^* a) \geq 0$$

Then, $\varphi$ is said to be *positive* & $(\mathcal{A}, \varphi)$ is called a *$*$-probability space* ($*$-ps).

**Def.** Let $(\mathcal{A},\varphi)$ be a $*$-ps. $a\in\mathcal{A}$ is

* self adjoint $\iff a=a^*$
* unitary $\iff a^*a=aa^*=1$
* normal $\iff a^*a=aa^*$

Recall:
A matrix is normal
$A^*A=AA^* \iff$
A has an orthonormal basis of eigenvectors

**Homework exercise.** Let $(\mathcal{A},\varphi)$ be a $*$-ps. Prove that

$$\varphi(a^*)=\overline{\varphi(a)} \qquad \forall a\in\mathcal{A}$$

(we say $\varphi$ is selfadjoint).

**Yet another HW exercise.** Prove the Cauchy-Schwartz inequality:

$$|\varphi(b^*a)|^2\leq\varphi(a^*a)\,\varphi(b^*b) \qquad \forall a,b\in\mathcal{A}$$

**Def.** Let $(\mathcal{A},\varphi)$ be a $*$-ps. $\varphi$ is faithful if

$$\varphi(a^*a) = 0 \implies a = 0$$

**Note.** If $\varphi$ is not faithful then $\exists a \in \mathcal{A}$ s.t.

$$\forall b \in \mathcal{A} \quad \varphi(ba) = 0$$

making $a$ degenerat wrt $\varphi$.

# Examples

**Example.** Complex random variables, having finite moments of all orders.

$$\varphi = \mathbb{E} : \mathcal{A} \longrightarrow \mathbb{C}$$
$$a \longmapsto \mathbb{E}[a]$$

$\mathcal{A}$ is a $*$-ps.

**Example.** $\mathcal{A} = M_d(\mathbb{C})$ - $*$-algebra of $d \times d$ complex matrices.

$\varphi = tr$ $\qquad tr(a) = \frac{1}{d} \sum_{i=1}^{d} \alpha_{ii}$ $\qquad a = (\alpha_{ij})$

normalized trace

can be thought of as $\mathbb{E}$ of the distribution which is uniform over the spectrum of $A$

**Example.** Let $G$ be a group. The group algebra $\mathbb{C}G$ is a $\mathbb{C}$-vector space having basis $G$

$$\mathbb{C}G = \left\{ \sum_{g\in G} \alpha_g g \;\middle|\; \alpha_g = 0 \text{ almost always} \right\}$$

* $$\left(\sum \alpha_g g\right)\left(\sum \beta_h h\right) = \sum_{g,h} \alpha_g \beta_h \, gh = \sum_{k\in G} \left(\sum_{gh=k} \alpha_g \beta_h\right) k$$

* $$\left(\sum \alpha_g g\right)^* = \sum \overline{\alpha_g}\, g^{-1}$$

$$\varphi = \tau_G : \mathbb{C}G \to \mathbb{C}$$

unit of $G$

$$\tau_G\left(\sum \alpha_g g\right) = \alpha_e$$

$*$-distribution in the analytic sense

Defthm. Let $(\mathcal{A},\varphi)$ be a $*$-ps.

Def & Theorem

Let $a \in \mathcal{A}$ normal. ← $aa^* = a^*a$

will formally define soon

If $\exists$ compactly supported probability measure $\mu$ on $\mathbb{C}$

s.t.
$$\int z^k \bar{z}^l \, d\mu(z) = \varphi(a^k (a^*)^l) \quad \forall k,l \in \mathbb{N}$$

then $\mu$ is uniquely determined, and is called

the *$*$-distribution of $a$ in the analytic sense.*

Uniqueness follows as we can approximate continuous functions using polynomials (Stone-Weierstrass).

# $*$-distribution of self-adjoint elements

## $*$-distribution of self adjoint elements.

$a = a^*$

Let $(\mathcal{A}, \varphi)$ be a $*$-p.s. $a \in \mathcal{A}$ self adjoint.

Suppose $a$ has a $*$-dist $\mu$. Then,

$$\int |z - \bar{z}|^2 \, d\mu(z) = \int (z - \bar{z})(\bar{z} - z) \, d\mu(z)$$

$$= \int (2 z \bar{z} - z^2 - \bar{z}^2) \, d\mu(z)$$

$$= 2\varphi(aa^*) - \varphi(a^2) - \varphi((a^*)^2)$$

$a = a^*$

$$= 0$$

But $z \longmapsto |z - \bar{z}|^2$ is a continuous non-negative function

$*$ measure theory alert

$\Longrightarrow \quad |z - \bar{z}|^2 = 0$ in $\mathrm{supp}(\mu)$

$\Longrightarrow \quad z - \bar{z} = 0$ in $\mathrm{supp}(\mu)$ $\quad \Longrightarrow \quad \mathrm{supp}(\mu) \subseteq \mathbb{R}$.

Example.

Let $a \in M_d(\mathbb{C})$ be a normal matrix with eigenvalues $\lambda_1, \dots, \lambda_d$.

Then,

$$\operatorname{tr}(a^k (a^*)^\ell) = \frac{1}{d} \sum_{i=1}^{d} \lambda_i^k \bar{\lambda}_i^\ell$$

$$= \int z^k \bar{z}^\ell \, d\mu(z)$$

want

where

$$\mu = \frac{1}{d} \sum_{i=1}^{d} \delta_{\lambda_i}$$

Dirac measure at $\lambda_i$

The eigenvalue distribution of the matrix $a$

we'll formalize it soon

$a = u^* D u$ unitary

$a^* = u^* D^\dagger u$

$a^k (a^*)^\ell = u^* D^k \bar{D}^\ell u$

$\Rightarrow \operatorname{tr}(a^k (a^*)^\ell) =$

$\operatorname{tr}(u^* D^k \bar{D}^\ell u) =$

$\operatorname{tr}(u u^* D^k \bar{D}^\ell) =$

$\operatorname{tr}(D^k \bar{D}^\ell) =$

$\frac{1}{d} \sum \lambda_i^k \bar{\lambda}_i^\ell$

Example.

Let $a \in M_d(\mathbb{C})$ be a normal matrix with eigenvalues $\lambda_1, \dots, \lambda_d$.

Then,

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The eigenvalue distribution of the matrix $a$

If $A$ & $B$ are two fixed normal matrices then the corresponding dist $\mu_A$ & $\mu_B$ are independent but that is not enough to determine $\mu_{A+B}$. Freeness, in this case, informally is about the eigenvectors! not the eigenvalues.

**Def.** Let $(\mathcal{A},\varphi)$ be a $*$-p.s.

$u \in \mathcal{A}$ is <u>Haar unitary</u> if it is unitary $(u^*u = uu^* = 1)$ &

$$\varphi(u^k) = 0 \qquad k \in \mathbb{Z} \setminus \{0\}.$$

The corresponding $*$-distribution is the "Haar measure" uniform over the circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$:

$$\varphi(u^k (u^*)^l) = \varphi(u^{k-l}) = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}.$$

$$\int_{\mathbb{T}} z^k \bar{z}^l \, dz = \int_0^{2\pi} e^{i(k-l)t} \frac{dt}{2\pi} \quad ''$$