additive

Finite free Convolution

Following "Finite free convolutions of polynomials"
by Marcus-Spielman-Srivastava

FPT Recap.

$(\mathcal{A}, \varphi)$

So that an analytic distribution exist

In FPT if a & b are free and normal then

This is actually the way we defined $\boxplus$ from freeness.

$$\mu_{a+b} = \mu_a \boxplus \mu_b$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$G_{a+b}(z) \qquad G_a(z) \qquad G_b(z)$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

Essentially inverses under compositions

$$G\Big(\underbrace{R(z)+\tfrac{1}{z}}_{K(z)}\Big) = z$$

$$R_{a+b}(z) = R_a(z) + R_b(z)$$

main theorem in FPT

Towards finite FPT

The very definition of $\boxplus$ required freeness & we saw that freeness manifest in infinite-dimensional operators only.

well, unless they are constant...

The idea in FFPT is to get some of the conclusion from FPT without freeness.

$$\mu_{a+b} = \mu_a \boxplus \mu_b$$

This is actually the way we defined $\boxplus$.

$$G_{a+b}(z) \qquad G_a(z) \qquad G_b(z)$$

Essentially inverses under compositions

$$G(\underbrace{R(z)+\tfrac{1}{z}}_{K(z)}) = z$$

$$R_{a+b}(z) = R_a(z) + R_b(z)$$

main theorem in FPT

Recall "the most important exercise" we solved in the recitation:

"If $\{a,b\}$ is free from a Haar u then a & ubu^* are free"

Thus,

well, μ_{ubu^*} but for a tracial φ, it's all the same

not necessarily free

* $\forall$ normal a & b

$$\mu_{a+ubu^*} = \mu_a \boxplus \mu_b$$

$$\Longrightarrow R_{a+ubu^*}(z) = R_a(z) + R_b(z)$$

Observation. Haar unitary finite-dimensional operators exist & the R-trasform can be defined for finite operators as well (as the inverse under composition of the Cauchy transform which surely exists). Can we mimic

* for finite-dimensional operators, namely define $\boxplus$ (finite) using Haar unitary (classically independent) & (freeness-free!) prove the R-transform equality?

also a little bit of a theorem

Definition. Let A, B be $d \times d$ complex normal matrices. Then,

characteristic polynomial

or Haar measure on $O(d)$

$$\mathop{\mathbb{E}}_{Q} \chi_x(A + QBQ^*)$$

The group of orthonormal $d \times d$ matrices

Haar measure on $U(d)$

The group of unitary $d \times d$ matrices

Depends only on $\mathrm{Spec}\, A$ & $\mathrm{spec}\, B$ (and not on the eigenvectors of A & B). We (can thus) define $\boxplus_d$ as the operator on $\mathbb{C}^{\leq d}[x]$ satisfying

polynomials of degree $\leq d$

$$\chi_x(A) \boxplus_d \chi_x(B) \triangleq \mathop{\mathbb{E}}_{Q} \chi_x(A + QBQ^*)$$

one may also consider the group of unitaries

$(U(n))$

Def. Let $O(n)$ denote the group of $n \times n$ orthogonal (unitary) matrices. The <u>Haar distribution</u> (on $O(n)$) is the unique distribution over $O(n)$ which is invariant under left & right multiplication with any $A \in O(n)$.

Existence & uniqueness is a theorem

won't be needed here

/orthogonal

Remark. Sampling a Haar unitary matrix can be done in several ways. E.g. Generate an $n \times n$ matrix Z with independent Gaussian random variables as entries, with mean 0 & variance 1. Then, apply Gram-Schmidt on Z-s columns.

Proof. A & B are normal so we can write

diagonal encode eigenvalues

$$A = V_A D_A V_A^* \quad \& \quad B = V_B D_B V_B^*$$

orthonormal encode eigenvectors

So $$\chi_x(A + QBQ^*) = \chi_x(V_A D_A V_A^* + Q V_B D_B V_B^* Q^*)$$

$$= \chi_x(V_A D V_A^* + \underbrace{V_A V_A^*}_{I} Q V_B D_B V_B^* Q^* \underbrace{V_A V_A^*}_{I})$$

$$= \chi_x(V_A(D + V_A^* Q V_B D_B V_B^* Q^* V_A) V_A^*)$$

$$= \chi_x(D + V_A^* Q V_B D_B V_B^* Q^* V_A)$$

cyclicity

$\chi_x(VEV^*) =$

$\chi_x(\underbrace{V^*V}_{I}E) =$

$\chi_x(E)$

So

$$\mathop{\mathbb{E}}_{Q} \chi_x(A+QBQ^*) = \mathop{\mathbb{E}}_{Q} \chi_x\Big(D_A + \underbrace{V_A^* Q V_B}_{\text{Haar uniform}} D_B \underbrace{V_B^* Q^* V_A}_{\text{conjugate}}\Big)$$

$$= \mathop{\mathbb{E}}_{Q} \chi_x(D_A + Q D_B Q^*)$$

∎

Now that we have the definition in place, let's move to the R-transform theorem.

The R- & K-transforms revisited

Def. For a polynomial p with roots $\lambda_1 \geq \cdots \geq \lambda_d$ may repeat we define the Cauchy transform

$$G_p(x) = \frac{1}{d} \sum_{i=1}^{d} \frac{1}{x - \lambda_i}$$

Remarks.

* our usual Cauchy transform

  $G_p(x) = G_\mu(x)$ where $\mu = \frac{1}{d} \sum_{i=1}^{d} \delta_{\lambda_i}$.

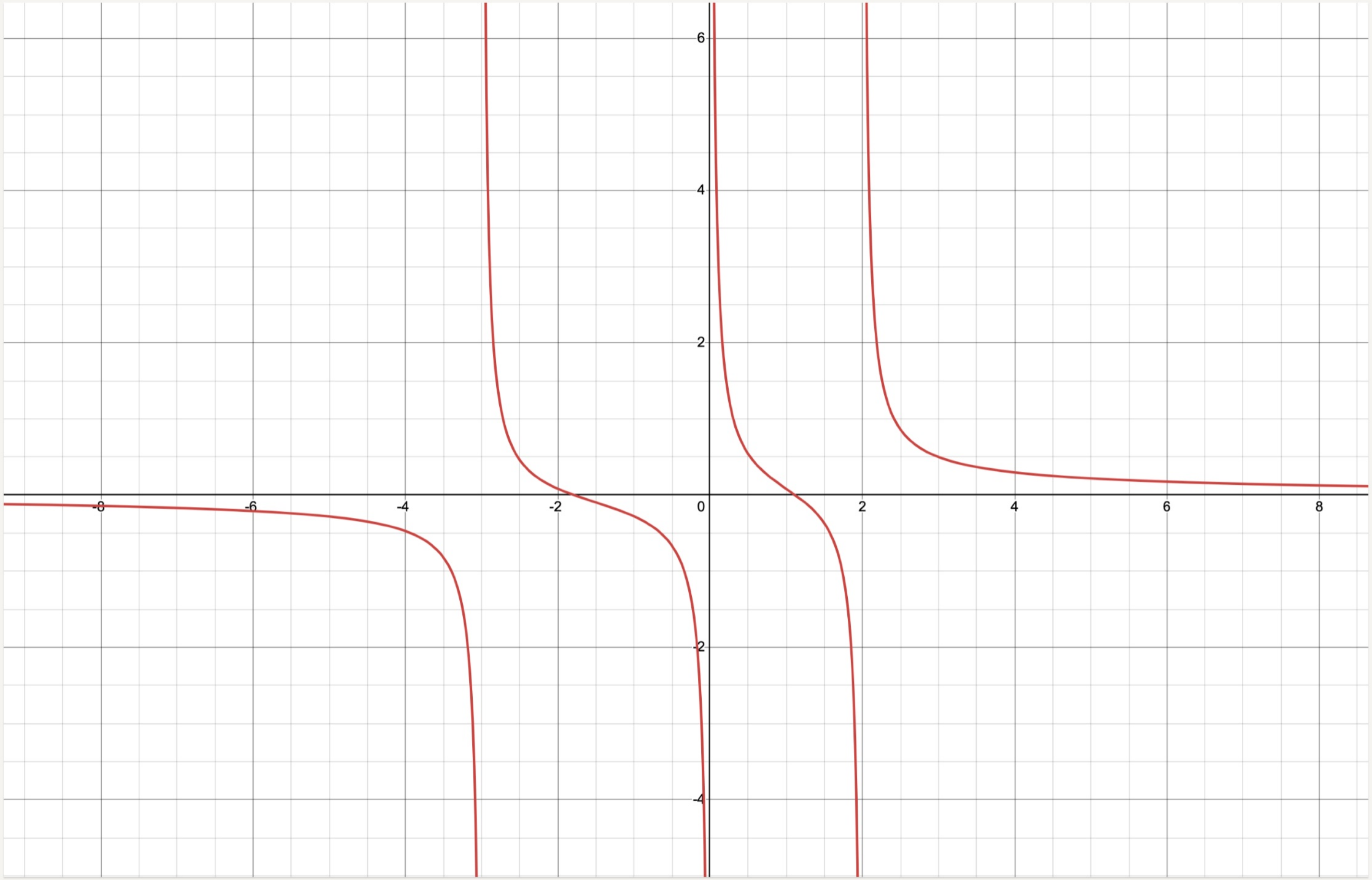
  Indeed,

  $$G_\mu(x) = \int \frac{1}{x-t}\, d\mu(t) = \frac{1}{d} \sum_{i=1}^{d} \frac{1}{x - \lambda_i} = G_p(x)$$

  Dirac measure

* $G_p(x) = \frac{1}{d} \frac{p'(x)}{p(x)}$

* In FPT we considered $G_\mu : \mathbb{C}^+ \to \mathbb{C}$. In finite FFPT we consider $G_\mu : \mathbb{R} \to \mathbb{R}$.



$$G_p(x) = \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} + \frac{1}{x+3}\right)$$

RRP

For a real-rooted polynomial $p(x) = c \cdot \prod_{i=1}^{d} x - \lambda_i$ ($c > 0$) the rightmost "branch", namely, $p|_{(\lambda_1, \infty)}$ has image $(0, \infty)$ & it is strictly monotone decreasing, so we can define a "max-inverse" for G_p.

Def. For an RRP p we define $K_p : (0, \infty) \longrightarrow \mathbb{R}$

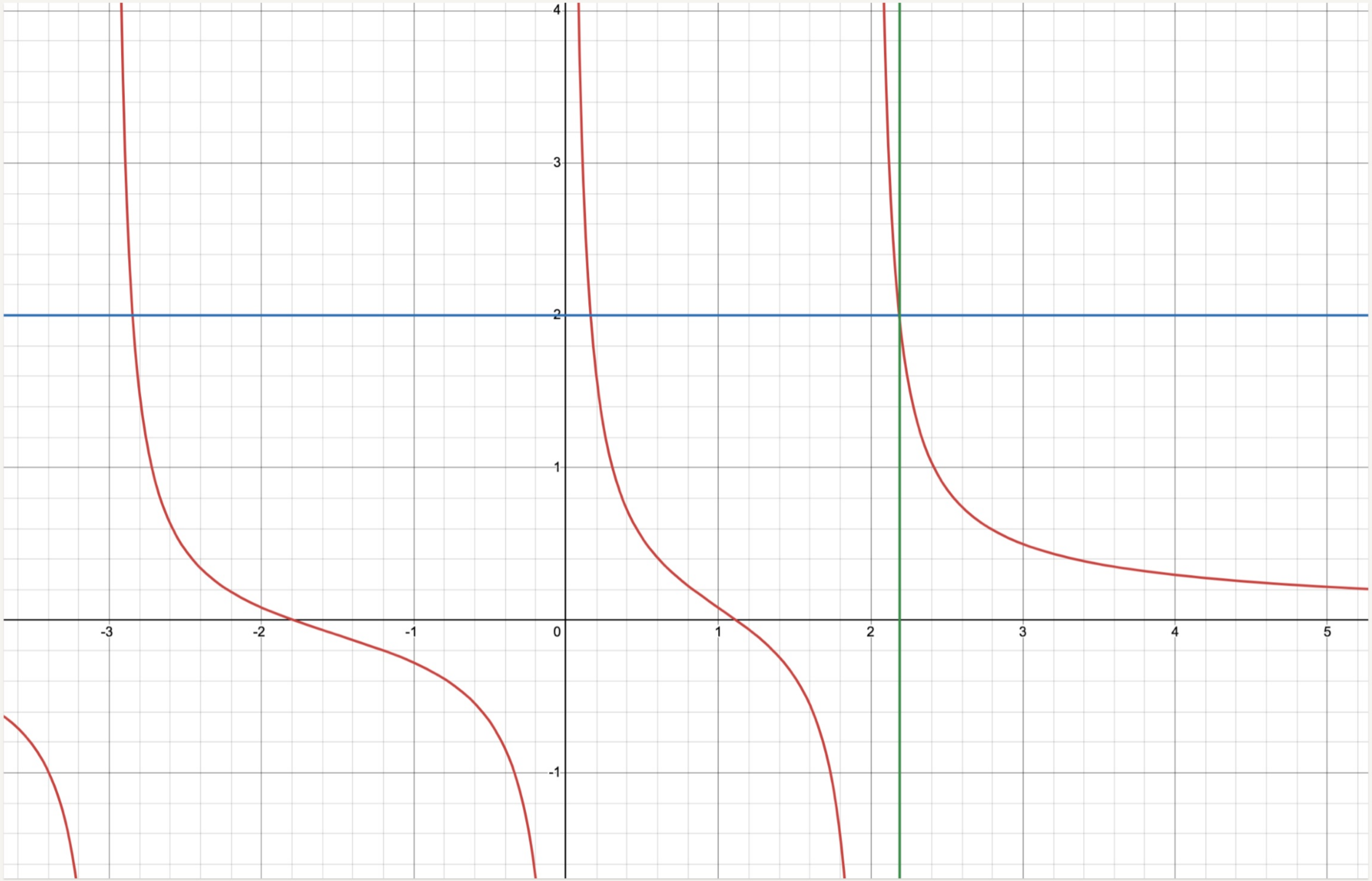
as follows:

$$K_p(w) = \max \{ x \mid G_p(x) = w \}$$

Remarks.

* $\forall w \in (0, \infty)$, $K_p(w) > \mathrm{maxroot}(p)$

* $\lim_{w \to \infty} K_p(w) = \mathrm{maxroot}(p)$



$K_p(2) \approx 2.19$

* The K-transform from FPT was defined to be the inverse as a formal power series of $G(z)$, though recall that we pick the solution (out of potentially) several solutions which is of the form

$$(*) \qquad \frac{1}{z} + k_1 + k_2 z + k_3 z^2 + \cdots$$

namely, when considering $G_p(x)$ for $p(t) = \prod (t-\lambda_i)$

this manifest itself in the special case in which $\mu = \frac{1}{d}\sum_{i=1}^{d} \delta_{\lambda_i}$ to the choice of the max root.

Indeed, it is the unique branch which has a form $\frac{1}{z} +$ [illegible] as in $(*)$.

In FPT we used $K_\mu(z)$ for computing the measure of sums. E.g.,

$$\begin{tikzcd} & dK_\mu(z) - \frac{d-1}{z} \arrow[d, phantom, "\parallel"] \\ K_\mu(z) \arrow[r] \arrow[d] & K_{\mu_d}(z) = R_{\mu_d}(z) + \frac{1}{z} \\ K_\mu(z) - \frac{1}{z} = R_\mu(z) \arrow[r] & R_{\mu_d}(z) = d\,R_\mu(z) \arrow[u] \end{tikzcd}$$

$$\mu_d = \underbrace{\mu \boxplus \cdots \boxplus \mu}_{d}$$

In FFPT, we won't invert $K_{p \boxplus q}$ to get $G_{p\boxplus q}$ but rather we'll use it to bound the largest (non-trivial) eigenvalue. Indeed,

$$\forall w \quad K_p(w) \geq \operatorname{maxroot}(p).$$

In particular, we ask for less information (querying only about the dist' support which is computationally simpler - no inversion from K back to G)

It will be useful to express $K_p(w)$ as the max root of a related polynomial $q = q(p, w)$:

As $G_p(x) = \frac{1}{d}\frac{p'(x)}{p(x)}$ we have

$$G_p(x) = w \iff p(x) - \frac{1}{wd}p'(x) = 0$$

$D \left(= \frac{\partial}{\partial x}\right)$ is the differentiation operator

Define the operator (on $\mathbb{C}[x]$)

Fancy way $U_\alpha = 1 - \alpha D$

$$U_\alpha p(x) = p(x) - \alpha D p(x)$$

Then,

$$G_p(x) = w \iff U_{\frac{1}{wd}} p(x) = 0$$

This only makes sense if $U_{\frac{1}{wd}} p$ is real-rooted -

see next page

So

$$K_p(w) = \max\{x \mid G_p(x) = w\} = \max \underline{\text{root}}\ U_{\frac{1}{wd}} p$$

we expressed the max-inverse of w as a max-root of a polynomial

Remarks

* As mentioned in the previous page, we have the following

Lemma. If p is real-rooted $\Rightarrow$ $\forall \alpha \geq 0$ $U_\alpha p$ is real-rooted.

This is not so difficult to prove and is left for you.

* What is less obvious is the following

Theorem ($\boxplus_d$ & real-rootedness). $\forall$ self adjoint $d \times d$ matrices A, B it holds that $\chi_x(A) \boxplus_d \chi_x(B)$ is real-rooted.

$$G(x) = \frac{1}{4}\left(\frac{1}{x-4} + \frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+3}\right)$$

$$p(x) = (x-4)(x-1)\,x\,(x+3)$$

$$\underbrace{\left(1-\frac{1}{4w}\right)}_{u_{\frac{1}{4w}}} p(x)$$

We also use the R-transform

$$R_p(w) = K_p(w) - \frac{1}{w}$$

Recall that in FPT $R_{\mu \boxplus \nu}(w) = R_\mu(w) + R_\nu(w)$. Here is a key result in FFPT which we'll turn to prove.

The FFPT R-Thm

Theorem. For $w > 0$ & RRPs p & q of degree exactly d, having positive leading coefficients

$$R_{p \boxplus_d q}(w) \leq R_p(w) + R_q(w)$$

namely, constant

Equality holds $\iff$ p or q takes the form $(x-\lambda)^d$.

The Pinching Lemma

The FFPT R-Theorem is proved by induction on d. The key technical tool supporting the induction is the so-called Pinching Lemma.

Def. For $d \geq 1$ we let $\mathbb{P}(d)$ denote the set of RRP of degree exactly d, having a positive leading coefficient.

Lemma. Let $p(x) \in \mathbb{P}(d)$ have at least two distinct roots.

Hence $d \geq 2$

Assume $p(x)$ is monic. Write

$$p(x) = \prod_{i=1}^{d} (x - \lambda_i) \quad \text{where} \quad \lambda_1 \geq \cdots \geq \lambda_d \ \& \ \lambda_1 > \lambda_k \ \text{for some } k$$

Then, $\forall \alpha > 0 \ \exists \mu, \rho \in \mathbb{R}$ s.t. $p(x) = \tilde{p}(x) + \hat{p}(x)$, where

pinched!

$$\tilde{p}(x) = (x-\mu)^2 \prod_{i \notin \{1,k\}} x - \lambda_i \quad \in \mathbb{P}(d) \quad \&$$

This factor is mistakenly omitted in the original paper, but it has no effect

$$\hat{p}(x) = \underbrace{(2\mu - (\lambda_1 + \lambda_k))}_{>0 \text{ as will be proven}} (x - \rho) \prod_{i \notin \{1,k\}} x - \lambda_i \quad \in \mathbb{P}(d-1)$$

Moreover,

mr from hereon

1 $\operatorname{maxroot}(U_\alpha \tilde{p}) = \operatorname{maxroot}(U_\alpha \hat{p}) = \operatorname{maxroot}(U_\alpha p)$

2 $\lambda_1 > \mu > \lambda_k$

3 $\rho > \lambda_1$. In particular, for $d \geq 3$ $\hat{p}$ has at least two distinct roots.

Note: $t > \lambda_1$ as $\operatorname{mr}(U_\alpha p)$, $\forall \alpha > 0$, is to the right of λ_1

pf. Let $t \triangleq \operatorname{mr}(U_\alpha p)$. Set

$$\mu \triangleq t - \frac{2}{\frac{1}{t-\lambda_1} + \frac{1}{t-\lambda_k}} \iff \frac{2}{t-\mu} = \frac{1}{t-\lambda_1} + \frac{1}{t-\lambda_k}$$

$\Updownarrow$

$t-\mu$ is the Harmonic mean of $t-\lambda_1$ & $t-\lambda_k$

Thus,

$$\frac{D\tilde{p}(t)}{\tilde{p}(t)} = \frac{2}{t-\mu} + \sum_{i \notin \{1,k\}} \frac{1}{t-\lambda_i} = \frac{Dp(t)}{p(t)}.$$

Recall $\quad G_p(x) = \omega \quad \Longleftrightarrow \quad p(x) - \frac{1}{\omega d} p'(x) = 0$

$$\Longleftrightarrow \quad U_{\frac{1}{\omega d}} p(x) = 0$$

so

$$\operatorname{mr}(U_\alpha p) = t \quad \Longleftrightarrow \quad t = \max \left\{ x \mid G_p(x) = \frac{1}{\alpha d} \right\}$$

$$\Longleftrightarrow \quad t = \max \left\{ x \mid \frac{Dp(x)}{p(x)} = \frac{1}{\alpha} \right\}$$

from previous page

so

$$\frac{D\tilde{p}(t)}{\tilde{p}(t)} = \frac{Dp(t)}{p(t)} = \frac{1}{\alpha}$$

Namely, t is one of the roots of $U_\alpha \tilde{p}$. We claim that $t = \operatorname{mr}(U_\alpha \tilde{p})$.

$$\frac{Dp(x)}{p(x)} = \frac{1}{x-\lambda_1} + \frac{1}{x-\lambda_k} + g(x) \qquad \frac{D\tilde{p}(x)}{\tilde{p}(x)} = \frac{2}{x-\mu} + g(x)$$

distinct as $\lambda_1 > \lambda_k$

As $t-\mu$ is the harmonic average of $t-\lambda_1$ & $t-\lambda_k$,

$$t-\mu \in (t-\lambda_1, t-\lambda_k) \Longrightarrow \mu \in (\lambda_k, \lambda_1)$$

which in particular implies (2).

we only "added" one root when moving from p to $\tilde{p}$ $\quad \mu < \lambda_1$

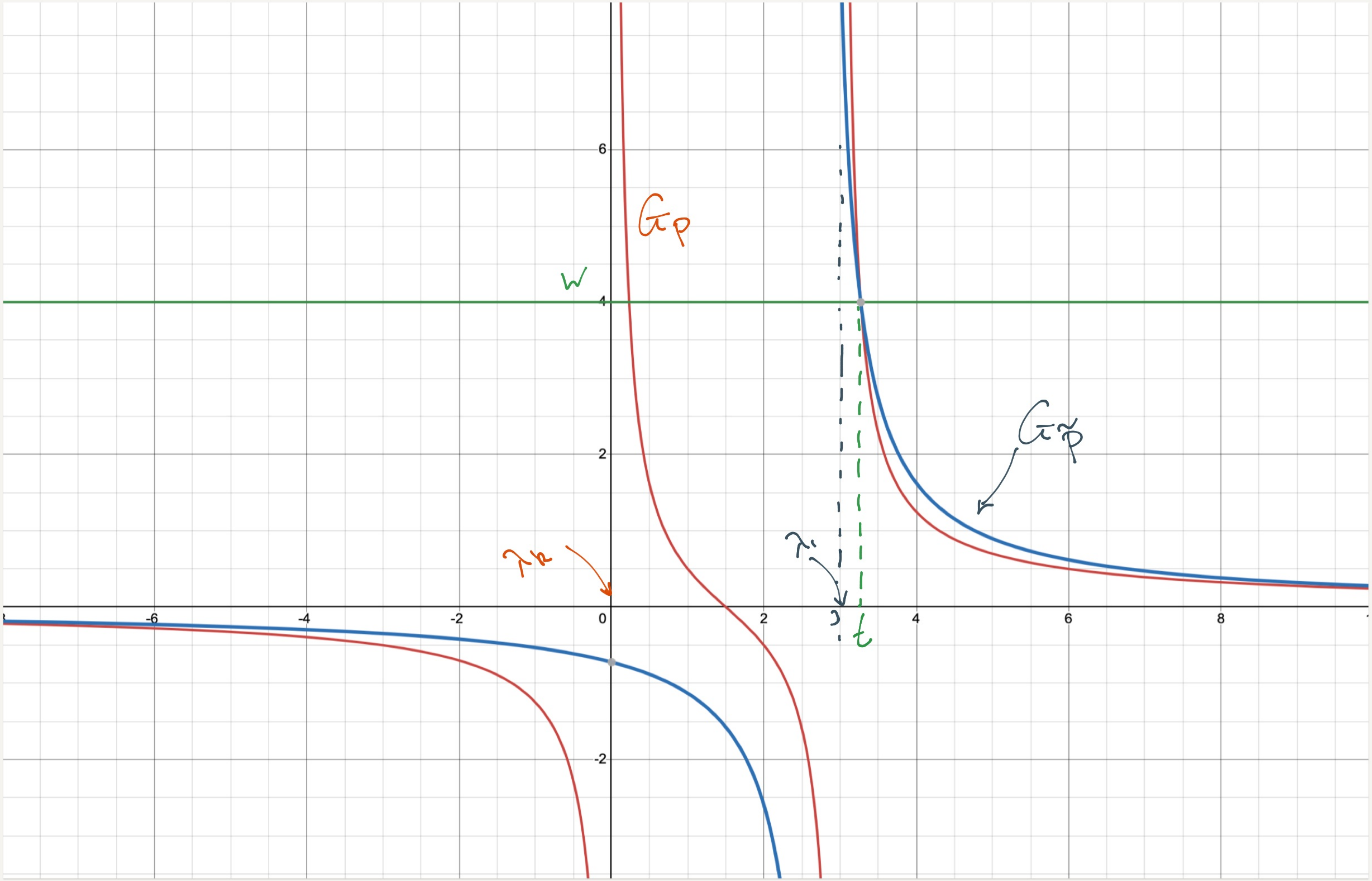
Now, $\quad mr(\tilde{p}) \leq \lambda_1$

$$\Longrightarrow \quad \frac{D\tilde{p}(x)}{\tilde{p}(x)} \text{ is strictly monotone decreasing in } (\lambda_1, \infty)$$

$> \lambda_1$

$$\Longrightarrow \quad \forall x > t \qquad \frac{D\tilde{p}(x)}{\tilde{p}(x)} < \frac{D\tilde{p}(t)}{\tilde{p}(t)} = \frac{1}{\alpha}$$

$$\Longrightarrow \quad mr(U_\alpha \tilde{p}) = t.$$



G_p
w
$G_{\tilde{p}}$
λ_k
λ_i
t
6
4
2
-2
-6
-4
0
8

Moving on to $\hat{p}(x)$, we have

$$\hat{p}(x) \overset{\triangle}{=} p(x) - \tilde{p}(x) = \underbrace{\left[(x-\lambda_1)(x-\lambda_k) - (x-\mu)^2\right]}_{\underbrace{(2\mu - (\lambda_1+\lambda_k))}_{>0}x - (\mu^2 - \lambda_1\lambda_k)} \cdot \prod_{i \notin \{1,k\}} x - \lambda_i$$

as $t-\mu = HM(t-\lambda_1, t-\lambda_k) < \frac{t-\lambda_1+t-\lambda_k}{2}$

$\Longrightarrow \hat{p}(x) \in \mathbb{P}(d-1)$

$\mathcal{U}_\alpha$ is a linear operator

Further, $\mathcal{U}_\alpha \hat{p}(t) = \mathcal{U}_\alpha p(t) - \mathcal{U}_\alpha \tilde{p}(t) = 0$

so it suffices to prove that $\mathrm{mr}\, \mathcal{U}_\alpha \hat{p} \le t$.

Because then, the strict monotonicity will close the deal

As in the $\tilde{p}$ case, it suffice to prove that $\mathrm{mr}\, \hat{p} < t$.

Now, the one root of $\hat{p}$ which is not a root of p is

$$\rho \triangleq \frac{-\mu^2 - \lambda_1\lambda_k}{2\mu - (\lambda_1+\lambda_k)}$$

But

$$t \overset{\Delta}{=} \mu - \frac{2}{\frac{1}{t-\lambda_1} + \frac{1}{t-\lambda_k}}$$

and so after ~~10~~ 20 minutes of calculations, we get

$$t-\rho = \frac{\overbrace{(\lambda_1-\mu)(\mu-\lambda_k)}^{>0 \text{ as } \lambda_1>\mu>\lambda_k}}{\underbrace{2\mu-(\lambda_1+\lambda_k)}_{>0 \text{ from orange bubble in previous slide}}} > 0$$

This proves 1 & 2. As for 3, namely $\rho > \lambda_1$:

$$\underbrace{\left(2\mu - (\lambda_1 + \lambda_k)\right)}_{>0}(\rho - \lambda_1) = \mu^2 - 2\lambda_1\mu + \lambda_1^2$$

$$= (\mu - \lambda_1)^2 > 0$$

$\Rightarrow \quad \rho > \lambda_1.$

Pinching Lemma

Preliminaries lemmata for the FFPT R-theorem

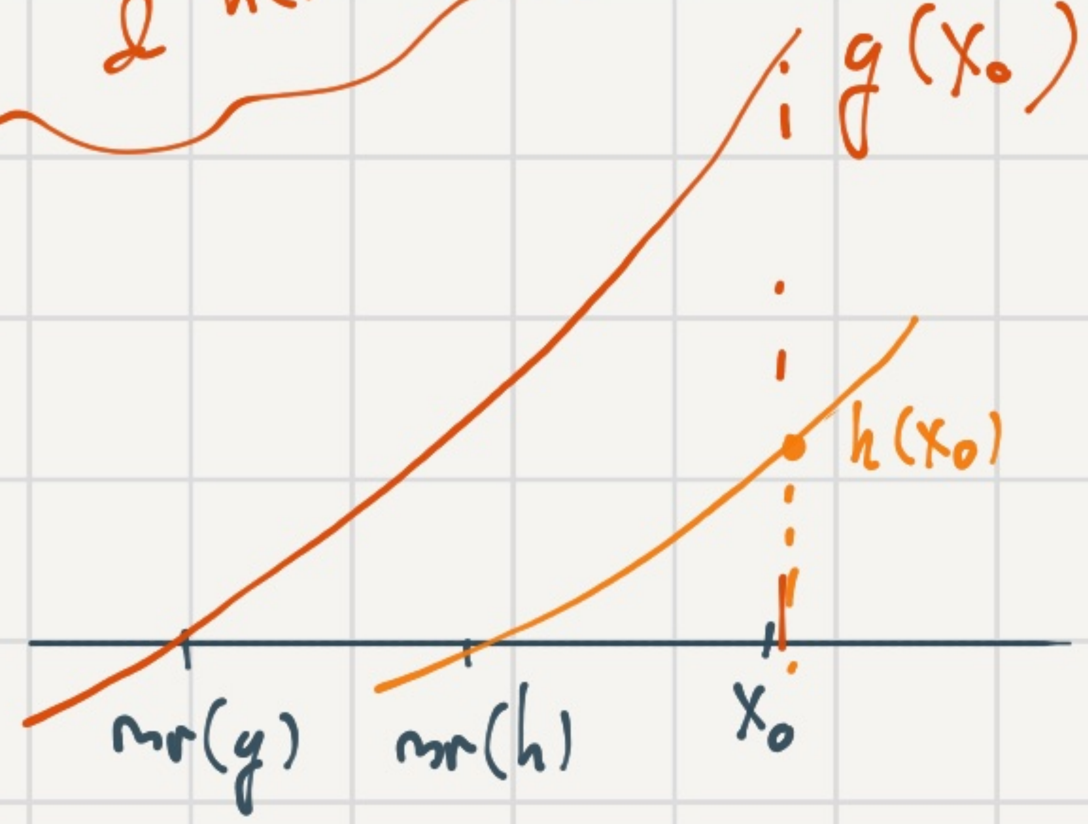
Little Lemma. Let g, h be RRP with positive leading coefficient s.t. $f = g+h$ is RRP. Then,

g,h RRP doesn't generally imply g+h is. E.g., $g(x) = (x-1)^2$ & $h(x) = x$

$$* \quad mr(f) \leq \max(mr(g), mr(h))$$

Moreover, equality holds iff

$$mr(f) = mr(g) = mr(h).$$



Pf. Denote $x_0 = mr(f)$. Assume by contradiction that $*$ is false. As g & h have positive leading coefficients $g(x_0)$ & $h(x_0) > 0 \implies 0 = f(x_0) = g(x_0) + h(x_0) > 0$ – contradiction. The "equality iff" part is left for you. ∎

Recall that $R_p(w) = K_p(w) - \frac{1}{w}$

$$= \max\{x \mid G_p(x) = w\} - \frac{1}{w}$$

$$= mr\left(U_{\frac{1}{wd}}\, p\right) - \frac{1}{w}$$

and so the theorem we wish to prove,

$$\forall w>0 \quad R_{p \boxplus_d q}(w) \le R_p(w) + R_q(w)$$

is equivalent to

$$\forall w>0 \quad mr\, U_{\frac{1}{wd}}(p \boxplus_d q) + \overset{\alpha d}{\frac{1}{w}} \le mr\left(U_{\frac{1}{wd}}\, p\right) + mr\left(U_{\frac{1}{wd}}\, q\right)$$

or

$$\boxed{\forall \alpha>0 \quad mr\, U_\alpha(p \boxplus_d q) + \alpha d \le mr(U_\alpha p) + mr(U_\alpha q)}$$

Lemma A. $\forall \alpha \geq 0,\ d \geq 2$ & $p \in \mathbb{P}(d)$

$$* \qquad \mathrm{mr}(U_\alpha D p) \leq \mathrm{mr}(U_\alpha p) - \alpha$$

with equality iff $p(x) = (x-\lambda)^d$.

pf. If $p(x) = (x-\lambda)^d$ then $Dp(x) = d(x-\lambda)^{d-1}$ and so

$$U_\alpha p = p(x) - \alpha D p(x) = (x-\lambda)^{d-1}(x - \lambda - \alpha d)$$

$$\Rightarrow \mathrm{mr}\, U_\alpha p \overset{\alpha, d \geq 0}{=} \lambda + \alpha d.$$

Similarly,

$$U_\alpha D p = Dp(x) - \alpha D^2 p(x) = d(x-\lambda)^{d-1} - \alpha d(d-1)(x-\lambda)^{d-2}$$

$$= d(x-\lambda)^{d-2}(x - \lambda - \alpha(d-1)) \quad \Rightarrow \quad \mathrm{mr}\, U_\alpha D p \overset{\alpha \geq 0,\ d \geq 1}{=} \lambda + \alpha(d-1).$$

even $d \geq 2$

which gives equality in $*$.

$$* \quad \mathrm{mr}(U_\alpha Dp) \leq \mathrm{mr}(U_\alpha p) - \alpha$$

We proceed by induction on $d \geq 2$

Base case $d = 2$.

Let $p(x) = (x-\lambda_1)(x-\lambda_2)$. Assume $\lambda_1 > \lambda_2$.

The case $\lambda_1 = \lambda_2$ was just handled

We compute

$$r \triangleq \alpha + \mathrm{mr}\, \underbrace{U_\alpha \underbrace{Dp}_{2x-(\lambda_1+\lambda_2)}}_{2x-(\lambda_1+\lambda_2)-2\alpha} = \alpha + \frac{\lambda_1+\lambda_2+2\alpha}{2} = 2\alpha + \frac{\lambda_1+\lambda_2}{2}.$$

whereas

$$U_\alpha p(x) = x^2 - (\lambda_1+\lambda_2)x + \lambda_1\lambda_2 - \alpha(2x-(\lambda_1+\lambda_2))$$

$$= x^2 - (\lambda_1+\lambda_2+2\alpha)x + \lambda_1\lambda_2 + \alpha(\lambda_1+\lambda_2)$$

Thus

$2\alpha + \frac{\lambda_1 + \lambda_2}{2}$

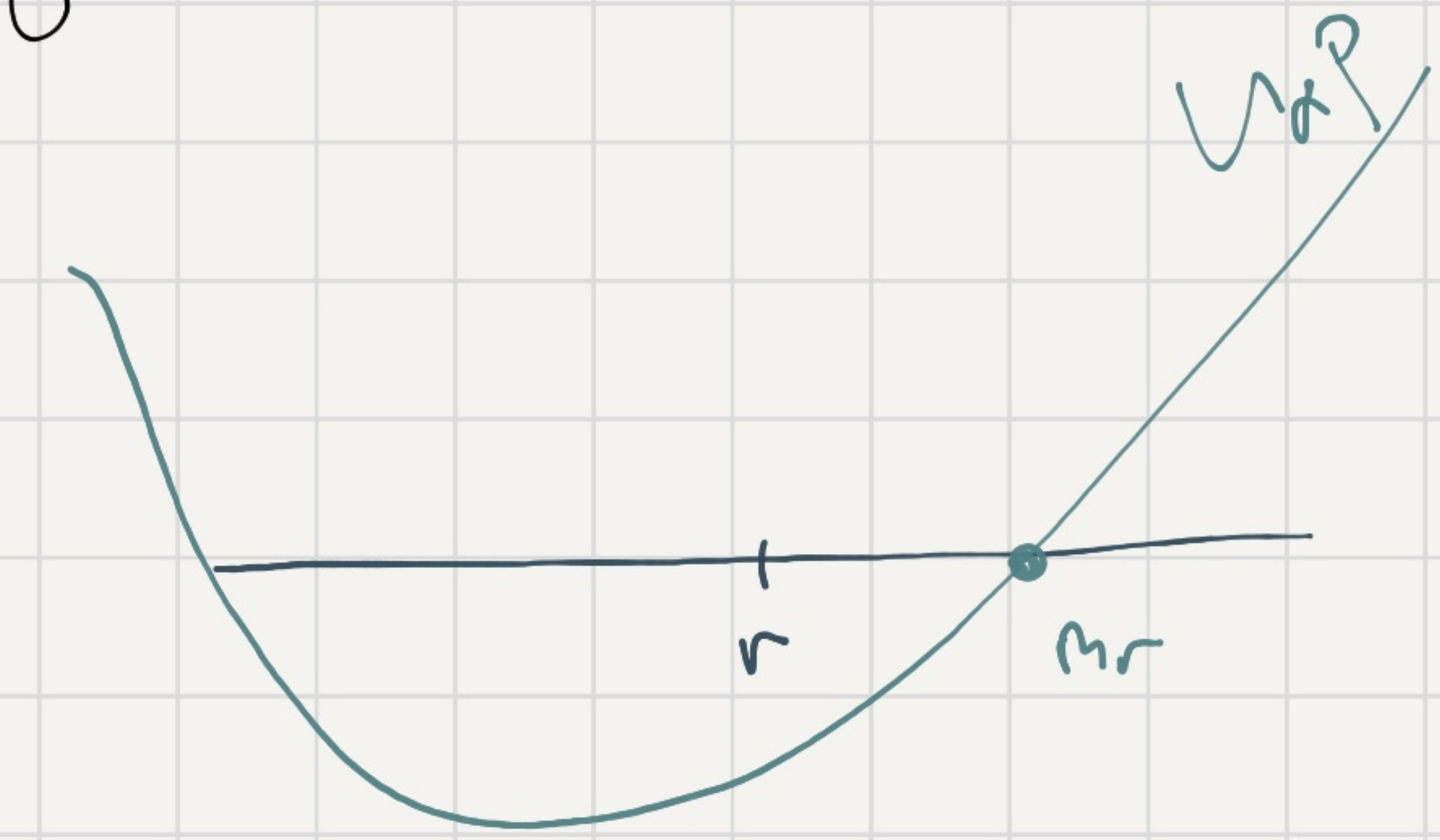
$$U_\alpha p(r) = r^2 - \overbrace{(\lambda_1 + \lambda_2 + 2\alpha)}^{2(r-\alpha)} r + \alpha(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2$$

$$= 2\alpha r - r^2 + \alpha \underbrace{(\lambda_1 + \lambda_2)}_{2r - 4\alpha} + \lambda_1 \lambda_2$$

$$= \underbrace{4\alpha r - r^2 - 4\alpha^2}_{-(r-2\alpha)^2 \,=\, -\left(\frac{\lambda_1 + \lambda_2}{2}\right)^2} + \lambda_1 \lambda_2$$

$$= -\left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 < 0$$

$\lambda_1 \neq \lambda_2$



But $U_\alpha p$ has a positive leading coefficient and so $m_r U_\alpha p > r$.

* $\mathrm{mr}(U_\alpha D p) \leq \mathrm{mr}(U_\alpha p) - \alpha$

Step. For RRP p, define

$$\phi(p) \triangleq \mathrm{mr}\, U_\alpha p - \alpha - \mathrm{mr}\, U_\alpha D p$$

we anyhow proved the lemma for a single-root poly

With this notation, we'll prove by induction on d that $\phi(p) > 0$ $\forall p \in \mathbb{P}(d)$ assuming p has at least 2 distinct roots.

strict!

Assume by contradiction $\exists\, p(x) \in \mathbb{P}(d)$ with at least two distinct roots s.t. $\phi(p) \leq 0$. WLOG p is monic.

ϕ doesn't care

Let $[-R,R]$ containing p-s roots & denote by $\mathbb{P}(d)[-R,R]$ all monic polynomials in $\mathbb{P}(d)$ having all their roots in $[-R,R]$.

$[-R,R]^d$ is compact

&

ϕ is a continuous function of p-s roots

$\Longrightarrow$ $\exists\, p_0 \in \mathbb{P}(d)[-R,R]$ at which ϕ attains its minimum

$\Downarrow$

$$\phi(p_0) \leq \phi(p) \leq 0$$

Note we may assume p_0 has at least two distinct roots. Indeed, if $\phi(p_0)<0$ that is indeed the case & if $\phi(p_0)=0$ we might as well As we proved the lemma for polynomials of the form $(x-\lambda)^d$ take $p_0 = p$.

By the pinching lemma $\exists\, \hat{p}$ & $\tilde{p}$ s.t. $p_0 = \tilde{p} + \hat{p}$ &

1) $\hat{p}$ & $\tilde{p}$ have positive leading coefficients.

2) $\deg \hat{p} = d-1$ & $d \geq 3 \implies \hat{p}$ has at least two distinct roots.

3) $\tilde{p} \in \mathbb{P}(d)[-R,R]$

4) $\operatorname{mr} U_\alpha \tilde{p} = \operatorname{mr} U_\alpha \hat{p} = \operatorname{mr} U_\alpha p_0$

By linearity,

$$U_\alpha D \tilde{p} + U_\alpha D \hat{p} = U_\alpha D p_0$$

<u>Claim.</u> $\operatorname{mr} U_\alpha D p_0 > \operatorname{mr} U_\alpha D \hat{p}$

Claim. $\operatorname{mr} U_\alpha D p_0 > \operatorname{mr} U_\alpha D \hat{p}$

pf. Otherwise,

$$\begin{aligned}
\phi(p_0) &= \operatorname{mr} U_\alpha p_0 - \alpha - \operatorname{mr} U_\alpha D p_0 \\
&\geq \operatorname{mr} U_\alpha \hat{p} - \alpha - \operatorname{mr} U_\alpha D \hat{p} \\
&= \phi(\hat{p}) > 0
\end{aligned}$$

otherwise

Pinching lemma

Induction $\hat{p} \in \mathbb{P}(d-1)$ has at least 2 distinct roots

In contradiction to $\phi(p_0) \leq 0$

Claim

$\blacksquare$

By Little Lemma

$$\operatorname{mr} U_\alpha D p_0 \leq \max\left(\operatorname{mr} U_\alpha D \tilde{p}, \operatorname{mr} U_\alpha D \hat{p}\right)$$

We claim that strict inequality holds, namely,

$$(**) \quad mr\, U_\alpha D p_0 < \max\left(mr\, U_\alpha D \tilde{p},\ mr\, U_\alpha D \hat{p}\right)$$

as otherwise the little lemma's moreover part contradicts the claim. Thus,

$$\begin{aligned}
\phi(p_0) &= \underbrace{mr\, U_\alpha p_0}_{\|} - \alpha - \underbrace{mr\, U_\alpha D p_0}_{\wedge} \\
&> \overbrace{mr\, U_\alpha \tilde{p}} - \alpha - \overbrace{mr\, U_\alpha D \tilde{p}} \\
&= \phi(\tilde{p})
\end{aligned}$$

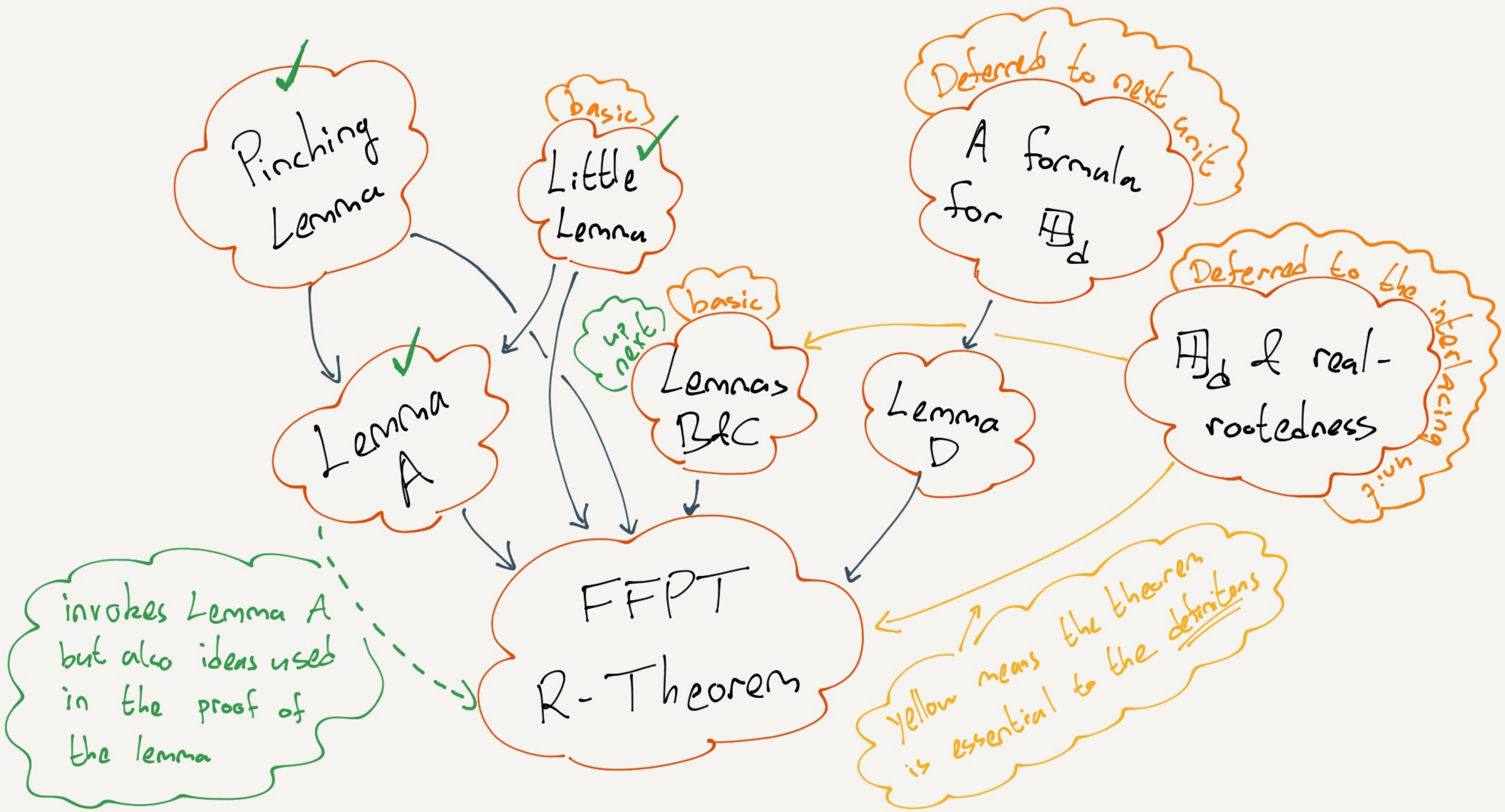
We just proved $(**)$ & the claim $mr\, U_\alpha D \hat{p} < mr\, U_\alpha D p_0$

$\tilde{p} \in \mathbb{P}(d)[-R, R]$

This stands in contradiction to the minimality of p_0.

Lemma

Recap & organization of the proof of the FFPT R-Theorem



Lemma B. $\forall \alpha \geq 0$, $q = (x-\lambda)^d$ $(\lambda \in \mathbb{R})$, $p \in \mathbb{P}(d)$.

$$* \quad \mathrm{mr}\, \mathcal{U}_\alpha (p \boxplus_d q) = \mathrm{mr}\, \mathcal{U}_\alpha p + \mathrm{mr}\, \mathcal{U}_\alpha q - \alpha d$$

pf. Let A be a normal matrix with $\chi_x(A) = p(x)$.

Let $B = \lambda I$. Then,

$$p \boxplus_d q\,(x) = \mathbb{E}_Q\, \chi_x(A + \overbrace{QBQ^*}^{\lambda I})$$

$$= \cancel{\mathbb{E}_Q}\, \chi_x(A + \lambda I) \qquad = \det(xI - (\lambda I + A))$$

$$= p(x - \lambda)$$

Thus,

$$\mathcal{U}_\alpha(p \boxplus_d q)(x+\lambda) = (\mathcal{U}_\alpha p)(x)$$

$$\Longrightarrow \qquad \mathrm{mr}\,\mathcal{U}_\alpha(p \boxplus_d q) = \mathrm{mr}\,\mathcal{U}_\alpha p + \lambda$$

Now,

$$(\mathcal{U}_\alpha q)(x) = \mathcal{U}_\alpha (x-\lambda)^d = (x-\lambda)^d - \alpha d\,(x-\lambda)^{d-1} = (x-\lambda)^{d-1}(x-\lambda-\alpha d)$$

and so $\mathrm{mr}\,\mathcal{U}_\alpha q = \lambda + \alpha d$, which indeed proves $\ast$.

Lemma B $\blacksquare$

Lemma C. If $p \in \mathbb{P}(d)$, $d \geq 3$, and Dp has just one distinct root then so does p.

pf. If $Dp(x) = a(x-\lambda)^{d-1} \implies p(x) = \frac{a}{d}(x-\lambda)^{d} + c$

but then if $c \neq 0$ (& $d \geq 3$) then $p(x)$ would have at least two complex roots. $\blacksquare$

Lemma D. Suppose $\deg p \leq d$ & $\deg q \leq d-1$. Then,

$$p(x) \boxplus_d q(x) = \left(\frac{1}{d} D p(x)\right) \boxplus_{d-1} q(x)$$

pf. We'll have to defer the proof for now $\ddot{-}$.

Proof of the FFPT

R- theorem

Recall our goal is to prove that

$$\forall_{\alpha>0} \quad \operatorname{mr}\,\mathcal{U}_\alpha(p \boxplus_d q) + \alpha d \le \operatorname{mr}(\mathcal{U}_\alpha p) + \operatorname{mr}(\mathcal{U}_\alpha q)$$

Lemma B proves the theorem if either p or q is of the form $(x-\lambda)^d$. Assume otherwise, and proceed by induction on $d \triangleq \max(\deg p, \deg q)$ to prove a stronger assertion:

namely, under the assumption that p & q each has at least two distinct roots

$$\operatorname{mr}\,\mathcal{U}_\alpha(p \boxplus_d q) + \alpha d < \operatorname{mr}\,\mathcal{U}_\alpha p + \operatorname{mr}\,\mathcal{U}_\alpha q$$

strict

<u>Base $d=1$.</u> Handled by Lemma B.

Assume $d \geq 2$ and fix q with at least two distinct roots.

Define

$$\phi(p) = \mathrm{mr}\, \mathcal{U}_\alpha p + \mathrm{mr}\, \mathcal{U}_\alpha q - d\alpha - \mathrm{mr}\, \mathcal{U}_\alpha (p \boxplus_d q)$$

q

As in the proof of Lemma A, assume by contradiction that $\exists\, p \in \mathbb{P}(d)$ with at least two distinct roots s.t. $\phi(p) \leq 0$.

monic wlog

Let $[-R, R]$ an interval containing its roots, and let p_0 be a minimizer of ϕ wrt $\mathbb{P}(d)[-R, R]$.

We may assume p_0 has at least two distinct roots: Indeed, if $\phi(p_0) < 0$ it is anyhow the case by Lemma B whereas if $\phi(p_0) \geq 0$ we can take $p_0 = p$.

By the Pinching Lemma, $\quad p_0 = \tilde{p} + \hat{p}$ where

1) roots of $\tilde{p} \in [-R, R] \implies \phi(\tilde{p}) \geq \phi(p_0)$

2) $\mathrm{mr}\, U_\alpha \tilde{p} = \mathrm{mr}\, U_\alpha \hat{p} = \mathrm{mr}\, U_\alpha p_0$

Note also that as U_α is linear & $\boxplus_d$ is bilinear

$$U_\alpha(p_0 \boxplus_d q) = U_\alpha(\hat{p} \boxplus_d q) + U_\alpha(\tilde{p} \boxplus_d q)$$

By Lemma D,

$$\mathrm{mr}\, U_\alpha(\hat{p} \boxplus_d q) = \mathrm{mr}\, U_\alpha(\hat{p} \boxplus_{d-1} Dq)$$

Recall q has at least 2 distinct roots. Hence, by Lemma C,
either Dq also has 2 distinct roots or $\deg q \leq 2$. should be handeled

Lemma D

$$\operatorname{mr} U_\alpha(\hat{p} \boxplus_d q) = \operatorname{mr} U_\alpha(\hat{p} \boxplus_{d-1} Dq)$$

Induction (in the first case)

$$\leq \underbrace{\operatorname{mr} U_\alpha \hat{p}}_{\substack{= \operatorname{mr} U_\alpha p_0 \\ \text{by } (2)}} + \underbrace{\operatorname{mr} U_\alpha Dq}_{\substack{< \operatorname{mr} U_\alpha q - \alpha \\ \text{by Lemma A}}} - (d-1)\alpha$$

Recall equality holds only if Dq has one distinct root, which is not the case here

$\phi(p_0) \leq 0$

$$< \operatorname{mr} U_\alpha p_0 + \operatorname{mr} U_\alpha q - d\alpha$$

$$\leq \operatorname{mr} U_\alpha(p_0 \boxplus_d q)$$

Recall

$$U_\alpha(p_0 \boxplus_d q) = U_\alpha(\hat{p} \boxplus_d q) + U_\alpha(\tilde{p} \boxplus_d q)$$

By Little Lemma (& in particular its moreover part),

$$\operatorname{mr} U_\alpha(\tilde{p} \boxplus_d q) > \operatorname{mr} U_\alpha(p_0 \boxplus_d q).$$

But $\operatorname{mr} U_\alpha \tilde{p} = \operatorname{mr} U_\alpha p$ and so $\phi(\tilde{p}) < \phi(p_0)$ – a contradiction.

$$\operatorname{mr} \mathcal{U}_\alpha(\tilde{p} \boxplus_d q) \;>\; \operatorname{mr} \mathcal{U}_\alpha(p_0 \boxplus_d q).$$

$$\phi(\tilde{p}) = \underbrace{\operatorname{mr} \mathcal{U}_\alpha \tilde{p}}_{\overset{\text{Pinching Lemma}}{=}\ \operatorname{mr} \mathcal{U}_\alpha p_0} + \operatorname{mr} \mathcal{U}_\alpha q - d\alpha - \underbrace{\operatorname{mr} \mathcal{U}_\alpha (\tilde{p} \boxplus_d q)}_{>\ \operatorname{mr} \mathcal{U}_\alpha\, p_0 \boxplus_d q}$$

$$< \phi(p_0)$$

As $\tilde{p} \in \mathbb{P}(d)[-R,R]$ we get a contradiction to the minimality at p_0.

Proof of finite R-transform $\blacksquare$

Recap & What's next?