Algebraic-Geometric Codes	**Fall 2024/5**
Problem Set 3	
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Problem 1. *Let K be a field with $char(K) \neq 2$, and let $F = K(x)[y]/\langle y^2 - f(x)\rangle$ where $f(T) \in K[T]$ and $\deg f = 2m+1 \geq 3$. Then $F/K(x)$ is a quadratic field extension. Show:*

(a) K is the full constant field of F.

(b) There is a exactly one prime divisor $\mathfrak{p} \in \mathbb{P}_F$ which is a pole of $x \in F$, and this $\mathfrak{p}$ is also the only pole of $y \in F$.

(c) If $r, s \in \mathbb{N}$ and $s < r - m$, then the elements $1, x, x^2, \ldots, x^r, y, xy, \ldots, x^s y$ are in $\mathcal{L}(2r\mathfrak{p})$.

(d) The genus g of F/K satisfies $g \leq m$.

Problem 2. *Let F/K be a function field with genus g.*

(a) Prove that if $x, y \in F \setminus K$ are such that $\deg(x)_\infty$ and $\deg(y)_\infty$ are coprime, then $F = K(x,y)$.

(b) Let $\mathfrak{p} \in \mathbb{P}_F$. Prove that for each $n \geq 2g$ there exists $x \in F$ with $(x)_\infty = n\mathfrak{p}$.

(c) Assume $g > 0$ and F/K has a degree one prime divisor. Prove that there exist $x, y \in F$ such that

$$[F : K(x)] = [F : K(y)] = 2g+1 \quad \text{and} \quad F = K(x,y).$$

Problem 3. *Let $\omega \in \Omega$ and $\alpha = (\alpha_\mathfrak{p}) \in \mathbb{A}$. Prove that $\omega_\mathfrak{p}(\alpha_\mathfrak{p}) = 0$ for almost all $\mathfrak{p} \in \mathbb{P}_F$, and*

$$\omega(\alpha) = \sum_{\mathfrak{p} \in \mathbb{P}} \omega_\mathfrak{p}(\alpha_\mathfrak{p}).$$

Hint: If $\omega \neq 0$, consider $S_1 := \operatorname{supp}((\omega))$, $S_2 := \{\mathfrak{p} \in \mathbb{P}_F \mid \nu_\mathfrak{p}(\alpha_\mathfrak{p}) < 0\}$ and $\beta \colon \mathbb{P}_F \to F$ given by

$$\beta_\mathfrak{p} := \begin{cases} \alpha_\mathfrak{p} & \mathfrak{p} \notin S_1 \cup S_2 \\ 0 & \textit{otherwise} \end{cases}.$$

Problem 4. *Consider the rational function field $\mathbb{C}(z)/\mathbb{C}$. Recall that we saw in class that the divisor $-2\mathfrak{p}_\infty$ is canonical. In fact, there exists a unique Weil differential $\omega \in \Omega$ with $(\omega) = -2\mathfrak{p}_\infty$ and $\omega_{\mathfrak{p}_\infty}(z^{-1}) = -1$.*

(a) Show that if $m \leq -2$ and $n \geq 0$ then

$$\omega_{\mathfrak{p}_\infty}((z-1)^m) = \omega_{\mathfrak{p}_1}((z-1)^n) = 0.$$

(b) Compute $\omega_{\mathfrak{p}_\infty}((z-1)^{-1})$*. Hint: Consider the function* $\frac{1}{z(z-1)}$.

(c) Using Problem 3, find $\omega_{\mathfrak{p}_\infty}((z-1)^k)$ *and* $\omega_{\mathfrak{p}_1}((z-1)^k)$ *for* $k \in \mathbb{Z}$.

(d) Compute $\omega_{\mathfrak{p}_1}\left(\frac{3z+2}{z^2-2z+1}\right)$.