Algebraic Geometric Codes

Recitation 05b

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Example $E = \mathbb{F}_q(x)[y]/\langle y^p + y - x^{p+1} \rangle$.

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We have the following diagram

$$\begin{array}{c} F \\ | \\ \mathbb{F}_q(x) \\ | \\ \mathbb{F}_q \end{array}$$

Where the first extension has $tr - deg$ of 1, and the second extension is algebraic.

F is a field, or, $y^p + y - x^{p+1}$ is irreducible

Proof.

From Gauss lemma is is enough to show that the polynomial is irreducible over $\mathbb{F}_q[x][y] \cong \mathbb{F}_q[x, y] \cong \mathbb{F}_q[y][x]$ and thus it is enough to show that the polynomial is irreducible in $\mathbb{F}_q[y][x]$. This follows from Eisenstein's criterion with $p = y$. □

Degree one palaces

We want to find all the degree one places in F. Note that P has degree one only if $P\mid_{\mathbb{F}_q(x)}$ has degree one (this is necessary but not sufficient). Recall that the degree one places in $F_q(x)$ correspond to the valuations

$$v_\infty \cup \{v_{x-\alpha} \mid \alpha \in F_q\}.$$

We need to consider extensions of these valuations.

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Therefore $p \mid v(x)$ denote $v(x) = -\alpha p$. It follows that $v(y) = -\alpha(p+1)$. Up to equivalence (why?) we can assume that $\alpha = 1$.
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$$deg(P) \cdot [\Gamma(v_\infty) : \Gamma(v)] \leq [F : \mathbb{F}_q(x)]$$

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$$deg(P) \cdot p \leq p \Rightarrow \deg(P) = 1.$$

Extensions of $v_{x-\alpha}$

For these valuations, we will consider the corresponding place:
$\varphi_\alpha : \mathbb{F}_q(x) \to \mathbb{F}_q : \quad \varphi_\alpha(x) = \alpha$.
We want to extend $\varphi : F \to L$, with $\varphi \mid_{\mathbb{F}_q(x)} = \varphi_\alpha$. It follows that

$$\varphi(y^p + y) = \varphi(x^{p+1}) = \alpha^{p+1} = N(\alpha).$$

Note that for every $\alpha \in \mathbb{F}_q, \alpha' = N(\alpha) \in \mathbb{F}_p$. More over, the equation $y^p + y = \alpha' \in \mathbb{F}_p$ has exactly p solutions in $\mathbb{F}_q$, i.e., there are p possible values for y in $\mathbb{F}_q$ such that $Tr(y) = \alpha'$. Each of these values corresponds to an exstention of φ_α, where $L = \mathbb{F}_q$.