# AGC Seminar

Gil Cohen

January 22, 2025

# General remarks

- The talks are partition to clusters which are mostly independent.
- Each student will email me its top 3 choices, in order, of a cluster.
- If you have a group set to take a cluster, please mention that as well as the name of your fellow students.
- I expect some coordination of talks within a cluster.
- You don't have to split the work according to sections, etc.
- Each student is expected to give a 30-35 minute talk.
- The talk must be well-organized and clear. You should welcome questions from the audience.

There are 26 registered students, and the projects are designated for 23-29 students.

# Overview


1. Kummer's Theorem
2. Differentials
3. Artin-Schreier extensions
4. Ramification
5. Some wild towers
6. Genus estimation

# Kummer's Theorem

Kummer's Theorem, which I stated in one of the examples video gives a way to determine the prime divisor lying about a given prime divisor.

Section 3.3, starting after Theorem 3.3.6.

This is a project for 2-3 students.

# Overview


1. Kummer's Theorem
2. Differentials
3. Artin-Schreier extensions
4. Ramification
5. Some wild towers
6. Genus estimation

# Differentials

Weil differentials, which were used for the proof of Riemann-Roch and Hurwitz Genus Formula, are very abstract. In this cluster we connect Weil differentials with "ordinary" differentials.

Sources:

1. Section 4.1 - Derivatives and differentials.
2. Section 4.2 - The $\mathfrak{p}$-adic completion.
3. Section 4.3 - Differentials and Weil differentials (here the Residue Theorem is proved).

This is a project for 5-6 students.

# Overview


1. Kummer's Theorem
2. Differentials
3. Artin-Schreier extensions
4. Ramification
5. Some wild towers
6. Genus estimation

# Artin-Schreier extensions

An Artin-Schreier extension is a degree $p$ extension of a field of characteristic $p$ - the extreme to Kummer extensions (in which the characteristic is coprime to the degree extension). A general form of such extension is $\mathsf{F} = \mathsf{E}(y)$ with

$$y^p - y = u.$$

(compared to $y^n = u$ in Kummer extensions).
These extensions are more complicated as they are not tame (they are called wild) extensions and so, e.g., Dedekind Different Theorem does not apply. However, they give many examples of optimal towers, including the famous Garcia-Sticthtenoth tower

$$y^p - y = \frac{x^p}{1 - x^{p-1}}.$$

# Artin-Schreier extensions

In this unit you'll cover the basic theory of Artin-Schreier extensions, and construct optimal wild towers.

Sources:

1. Section 3.7, starting from Lemma 3.7.7 - Galois extensions I
2. Section 6.4 - Some elementary $p$-extensions.

This is a project for 3-4 students.

# Overview

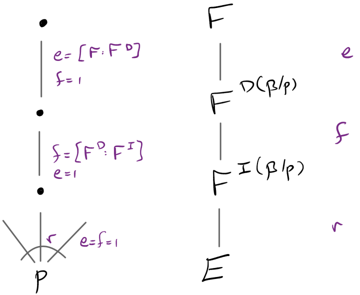

1. Kummer's Theorem
2. Differentials
3. Artin-Schreier extensions
4. Ramification
5. Some wild towers
6. Genus estimation

# Ramification

Recall that in Galois extensions we have

$$efr = p.$$

In this cluster we will dig deeper into this, ramification, and the different exponent. E.g., we will decompose F/E to



$$\mathrm{D}(\mathfrak{P}/\mathfrak{p}) = \{\sigma \in G \mid \sigma\mathfrak{P} = \mathfrak{P}\},$$
$$\mathrm{I}(\mathfrak{P}/\mathfrak{p}) = \ker(\text{something}).$$

# Ramification

We will further consider a more refined filtration via ramification groups.

It is easy to prove that

$$\mathrm{D}(\mathfrak{P}/\mathfrak{p}) = \{\sigma \in G \mid \forall z \in \mathcal{O}_{\mathfrak{P}} \quad \upsilon_{\mathfrak{P}}(\sigma z - z) \geq 0\}.$$

One can further show that

$$\mathrm{I}(\mathfrak{P}/\mathfrak{p}) = \{\sigma \in G \mid \forall z \in \mathcal{O}_{\mathfrak{P}} \quad \upsilon_{\mathfrak{P}}(\sigma z - z) \geq 1\}.$$

So it is natural to consider

$$G_i(\mathfrak{P}/\mathfrak{p}) = \{\sigma \in G \mid \forall z \in \mathcal{O}_{\mathfrak{P}} \quad \upsilon_{\mathfrak{P}}(\sigma z - z) \geq i + 1\}.$$

It turns out that $G_{i+1} \triangleleft G_i$ and that $G_i / G_{i+1}$ is isomorphic to an additive subgroup of the residue field $\mathrm{F}_{\mathfrak{P}}$, hence is an elementary abelian $p$-group of exponent $p$.

# Ramification

Based on these results some important theorems are proven, e.g., Hilbert's Different Formula

$$d(\mathfrak{P}/\mathfrak{p}) = \sum_{i=0}^{\infty} (|G_i(\mathfrak{P}/\mathfrak{p})| - 1),$$

as well as Abhyankar's Lemma and other results we used about ramification in compositum.

This is a beautiful and insightful cluster that involves Galois theory and some nice group theory.

# Ramification

Sources:

1. Section 3.8 - Galois extensions II.
2. Section 3.9 Ramification and splitting in compositum of function fields.

This is a project for 5-6 students.

# Overview


1 Kummer's Theorem

2 Differentials

3 Artin-Schreier extensions

4 Ramification

5 Some wild towers

6 Genus estimation

# Some wild towers

In this cluster you will present several examples of optimal wild towers (all of which are Artin-Schreier extensions), including the famous Garcia-Sticthtenoth tower

$$y^p - y = \frac{x^p}{1 - x^{p-1}}.$$

Based Section 7.4. A project for 5-6 students.

# Overview

1 Kummer's Theorem

2 Differentials

3 Artin-Schreier extensions

4 Ramification

5 Some wild towers

6 Genus estimation

# Genus estimation

Some general bounds on the genus: Castelnuovo's inequality and (as a special case) Riemann's inequality.

1. Section 3.10 - Inseparable extensions.
2. Section 3.11 - Estimates for the genus of function fields.

This is a project for 3-4 students.