Free Cumulants

Following Nica-Speicher Chapter 11

Def. Let $\mathcal{A}$ be a unital algebra. Given a sequence

$$\forall n \geq 1 \quad \rho_n : \mathcal{A}^n \to \mathbb{C}$$

$$(a_1, \dots, a_n) \mapsto \rho_n(a_1, \dots, a_n)$$

of multilinear functionals, we extend (ρ_n) to a family of multilinear functionals

$$\forall n \geq 1 \ \forall \pi \in NC(n) \quad \rho_\pi : \mathcal{A}^n \to \mathbb{C}$$

$$(a_1, \dots, a_n) \mapsto \rho_\pi[a_1, \dots, a_n]$$

as follows: If $\pi = \{V_1, \dots, V_r\} \in NC(n)$ then

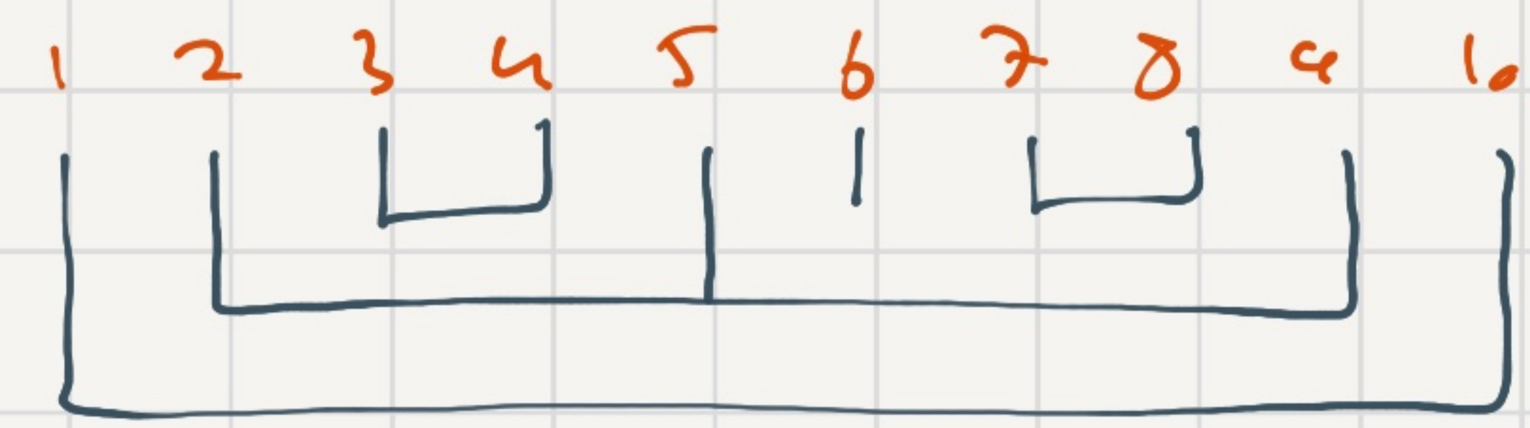
$$\rho_\pi[a_1, \dots, a_n] \triangleq \rho(V_1)[a_1, \dots, a_n] \cdots \rho(V_r)[a_1, \dots, a_n]$$

where if $V = \{i_1 < \dots < i_s\}$ then $\rho(V)[a_1, \dots a_n] \triangleq \rho_s(V_{i_1}, \dots, V_{i_s})$

(ρ_π) is called **the multiplicative family of functions on NC** determined by (ρ_n).

Note. This is indeed an extension as $\rho_{1_n}[a_1, \dots, a_n] = \rho_n(a_1, \dots, a_n)$.

Example. $\pi =$



$$\rho_\pi[a_1, \dots, a_{10}] = \rho_2(a_1, a_{10}) \cdot \rho_3(a_2, a_5, a_9) \cdot \rho_2(a_3, a_4) \cdot \rho_1(a_6) \cdot \rho_2(a_7, a_8).$$

Def. Let $\mathcal{A}$ be a unital algebra and $\varphi: \mathcal{A} \to \mathbb{C}$ a unital linear functional. For $n \geq 1$ we define

$$\varphi_n : \mathcal{A}^n \longrightarrow \mathbb{C}$$
$$(a_1, \dots, a_n) \longmapsto \varphi(a_1 \cdots a_n)$$

we exted this to NC by

$$\forall n \geq 1 \quad \forall \pi \in NC(n) \qquad \varphi_\pi [a_1, \dots, a_n] = \prod_{V \in \pi} \varphi(V) [a_1, \dots, a_n].$$

Recap.

ρ $\qquad$ $\rho : \mathcal{A} \to \mathbb{C}$

$\rightsquigarrow$

$\forall n \geq 1$ $\qquad$ ρ_n $\qquad$ $\rho_n : \mathcal{A}^n \mapsto \mathbb{C}$

$\rho_n(a_1, \dots, a_n) = \rho(a_1 \cdots a_n)$

$\rightsquigarrow$

$\forall n \geq 1$ $\quad$ $\forall V \subseteq [n]$ $\qquad$ $\rho(V)$ $\qquad$ $\rho(V) : \mathcal{A}^n \to \mathbb{C}$ $\qquad$ $V = \{i_1 < \dots < i_s\}$

$\rho(V)[a_1, \dots a_n] = \rho_s(a_{i_1}, \dots, a_{i_s})$

$\rightsquigarrow$

$\forall n \geq$ $\quad$ $\forall \pi \in NC(n)$ $\qquad$ ρ_π $\qquad$ $\rho_\pi : \mathcal{A}^n \to \mathbb{C}$

$\rho_\pi[a_1, \dots, a_n] = \prod_{V \in \pi} \rho(V)[a_1, \dots, a_n]$

Def. Let $(\mathcal{A}, \varphi)$ be a ncps. The corresponding free cumulants

$(\kappa_\pi)_{\pi \in NC}$ are:

n is s.t. $\pi \in NC(n)$

$$\kappa_\pi : \mathcal{A}^n \longrightarrow \mathbb{C}$$

$$(a_1, \dots, a_n) \longmapsto \kappa_\pi[a_1, \dots, a_n]$$

which are defined as

$$\kappa_\pi[a_1, \dots, a_n] \triangleq \sum_{\sigma \leq \pi} \varphi_\sigma[a_1, \dots, a_n] \, \mu(\sigma, \pi).$$

We also denote κ_{1_n} by κ_n.

The following result (essentially) follows from previous lectures.

Proposition.

1) The free cumulants κ_π is a multiplicative family of functions, namely,

$$\kappa_\pi[a_1,\dots,a_n] = \prod_{V\in\pi} \kappa(V)[a_1,\dots,a_n]$$

2) In particular (κ_n) encodes all information in (κ_π).

Moreover, an equivalent def to (κ_π) is obtained by defining (κ_n) and extend multiplicatively.

see next page

$$\begin{tikzcd}
 & \rho \arrow[d, squiggly] & \\
\forall n\geq 1 & \rho_n \arrow[d, squiggly] & \kappa_{1_n} =: \kappa_n : \mathcal{A}^n \to \mathbb{C} \arrow[d] \\
\forall n\geq 1 \ \forall V\subseteq[n] & \rho(V) \arrow[d, squiggly] & \kappa(V) \arrow[d, squiggly] \\
\forall n\geq \ \forall \pi\in NC(n) & \rho_\pi \arrow[r] & \kappa_\pi \arrow[uu, bend right]
\end{tikzcd}$$

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$$\kappa_n(a_1,\dots,a_n) = \sum_{\sigma\in NC(n)} \varphi_\sigma[a_1,\dots,a_n]\,\mu(\sigma,1_n). \qquad (*)$$

3) A third equivalent definition of (κ_π) is as a multiplicative family s.t. $\forall n\geq 1$ $\forall\, a_1,\dots,a_n\in\mathcal{A}$,

$$\varphi_n(a_1,\dots,a_n)=\varphi(a_1\cdots a_n) = \sum_{\pi\in NC(n)} \kappa_\pi[a_1,\dots,a_n] \qquad (**)$$

Proof sketch.

The proof can be adjusted to work in this more general setting

Up to the appearence of the variables $a_1,\dots,a_n$, that (κ_n) is multiplicative follows since (μ_n) & (φ_n) are multiplicative.

The equivalence in item 3 is due to the Mobius inversion formula, again, extended to facilitate the a_i-s. ∎

Def. Equations $*$ & $**$ are called the <u>moment-cumulant formulas</u>.

$$\kappa_1(a_1) = \sum_{\sigma \in NC(1)} \varphi_\sigma[a_1]\,\mu(\sigma, 1_n)$$

$\underline{\kappa_1}$.

$$\kappa_1(a_1) = \varphi(a_1)$$

$\underline{\kappa_2}$.

$$\kappa_2(a_1, a_2) = \varphi_{||}[a_1, a_2]\,\underbrace{\mu(||, \sqcup)}_{-1} + \varphi_{\sqcup}[a_1, a_2]\,\underbrace{\mu(\sqcup, \sqcup)}_{1}$$

$$= \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$$

$\underline{K_3}$..

$$
\begin{aligned}
K_3(a_1,a_2,a_3) = {} & \varphi_{|||}[a_1,a_2,a_3]\,\mu\underbrace{(|||,\text{ш})}_{2} + \\
& \varphi_{|\sqcup}[a_1,a_2,a_3]\,\mu\underbrace{(|\sqcup,\text{ш})}_{-1} + \\
& \varphi_{\dot\sqcup}[a_1,a_2,a_3]\,\mu\underbrace{(\dot\sqcup,\text{ш})}_{-1} + \\
& \varphi_{\sqcup|}[a_1,a_2,a_3]\,\mu\underbrace{(\sqcup|,\text{ш})}_{-1} + \\
& \varphi_{\text{ш}}[a_1,a_2,a_3]\,\mu\underbrace{(\text{ш},\text{ш})}_{1}
\end{aligned}
$$

$$
\begin{aligned}
= {} & \varphi(a_1a_2a_3) - \varphi(a_1a_2)\varphi(a_3) - \varphi(a_1a_3)\varphi(a_2) \\
& - \varphi(a_2a_3)\varphi(a_1) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3).
\end{aligned}
$$

Proposition. Let $(\kappa_n)_{n\geq 1}$ be the cumulants corresponding to φ.

Then,

$$\varphi \text{ is a homomorphism} \iff \forall n\geq 2 \quad \kappa_n = 0.$$

Pf. $\Rightarrow$ If κ_2 vanish then

$$\forall a_1, a_2 \in \mathcal{A} \qquad 0 = \kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$$

Namely, φ is a homomorphism.

$\Leftarrow$
$$\varphi_\sigma[a_1, \dots, a_n] = \prod_{V\in\sigma} \underbrace{\varphi(V)[a_1, \dots, a_n]}_{\substack{\varphi(a_{i_1}\cdots a_{i_s}) = \varphi(a_{i_1})\cdots\varphi(a_{i_s}) \\ V = \{i_1 < \cdots < i_s\}}} = \varphi(a_1)\cdots\varphi(a_n)$$

So

$$K_n = \sum_{\sigma \in NC(n)} \varphi_\sigma[a_1, \dots, a_n]\, \mu(\sigma, 1_n)$$

$$= \varphi(a_1) \cdots \varphi(a_n) \underbrace{\sum_\sigma \overbrace{\mu}^{\zeta(0_n,\sigma)}(\sigma, 1_n)}_{(\zeta * \mu)(0_n, 1_n) = \delta(0_n, 1_n) = 0}$$

$n \geq 2$

∎

really of proof

Corollary. $K_n(1, 1, \dots, 1) = \begin{cases} 1 & n = 1 \\ 0 & n \geq 2 \end{cases}$

Product as arguments

especially associativity

How does the multiplicative $\underbrace{\text{structure}}$ of $\mathcal{A}$ reflects in (κ_n)?

For example,

$$\begin{aligned}\varphi_2(a_1a_2, a_3) &= \varphi((a_1a_2)a_3) \\ &= \varphi(a_1(a_2a_3)) = \varphi_2(a_1, a_2a_3)\end{aligned}$$

but it's not generally true that

$$\kappa_2(a_1a_2, a_3) = \kappa_2(a_1, a_2a_3)$$

$$\begin{array}{ccc} \| & & \| \\ \varphi(a_1a_2a_3) - \varphi(a_1a_2)\varphi(a_3) & = & \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) \end{array}$$

$$\Updownarrow$$

$$\varphi(a_1a_2)\varphi(a_3) = \varphi(a_1)\varphi(a_2a_3)$$

We've looking at $a_1, \dots, a_n \in \mathcal{A}$, $1 \le i(1) < i(2) < \dots < i(m) = n$

and

$$\kappa_\tau[a_1 \cdots a_{i(1)}, a_{i(1)+1} \cdots a_{i(2)}, \dots, a_{i(m-1)+1} \cdots a_{i(m)}]$$

for some $\tau \in NC(m)$.

Goal! Express in terms of $(\kappa_\pi[a_1, \dots, a_n])_{\pi \in NC(n)}$

Notation. We define the embedding

Depending on $i(1), \dots, i(m)$

$$\hat{\cdot} : NC(m) \longrightarrow NC(n)$$
$$\tau \mapsto \hat{\tau}$$

as follows: $\hat{\tau}$ is obtained from τ by replacing each $j \in [m]$ by

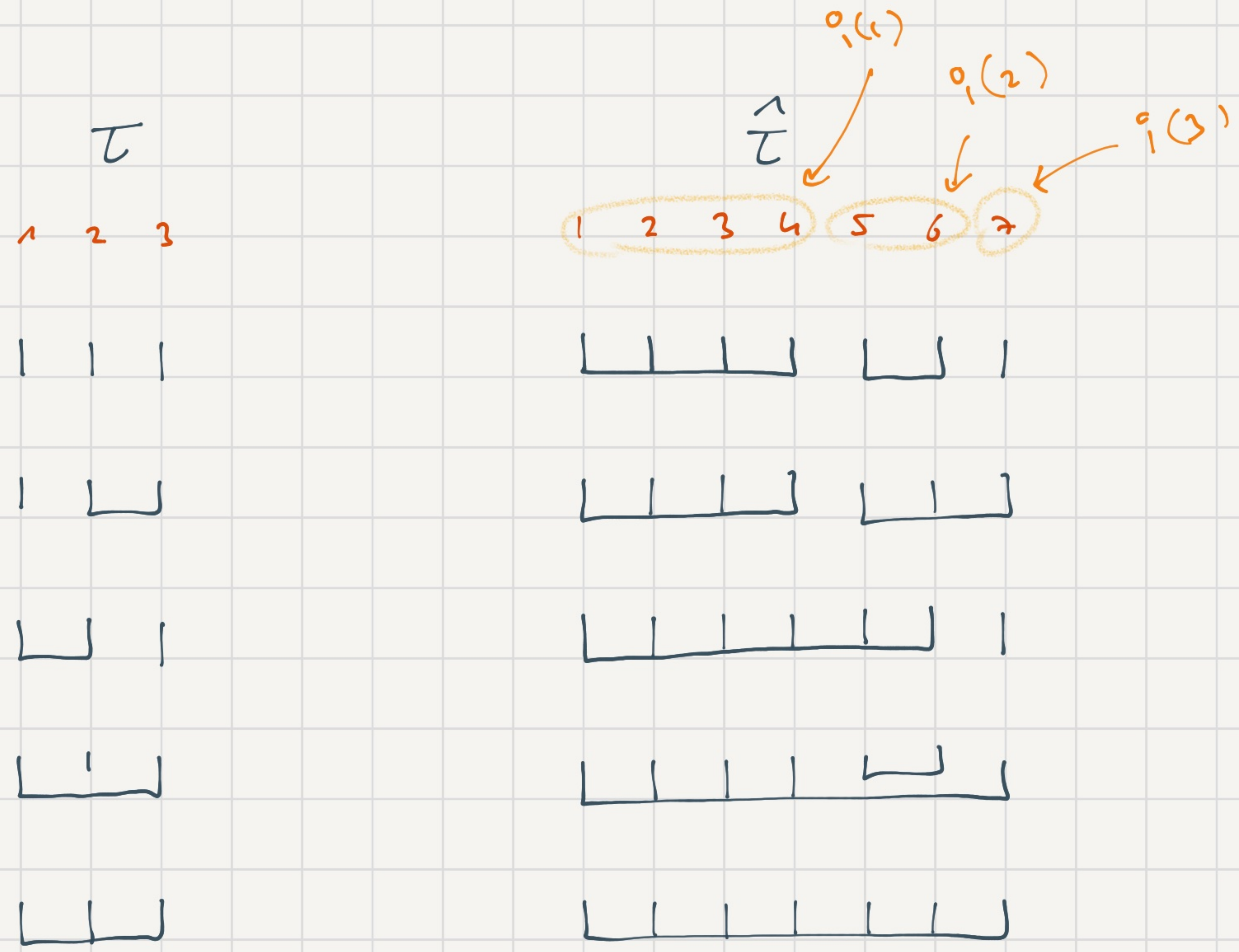
$i(j-1)+1, \dots, i(j)$ s.t. $i(j-1)+1 \sim_{\hat{\tau}} i(j-1)+2 \sim_{\hat{\tau}} \cdots \sim_{\hat{\tau}} i(j)$ &

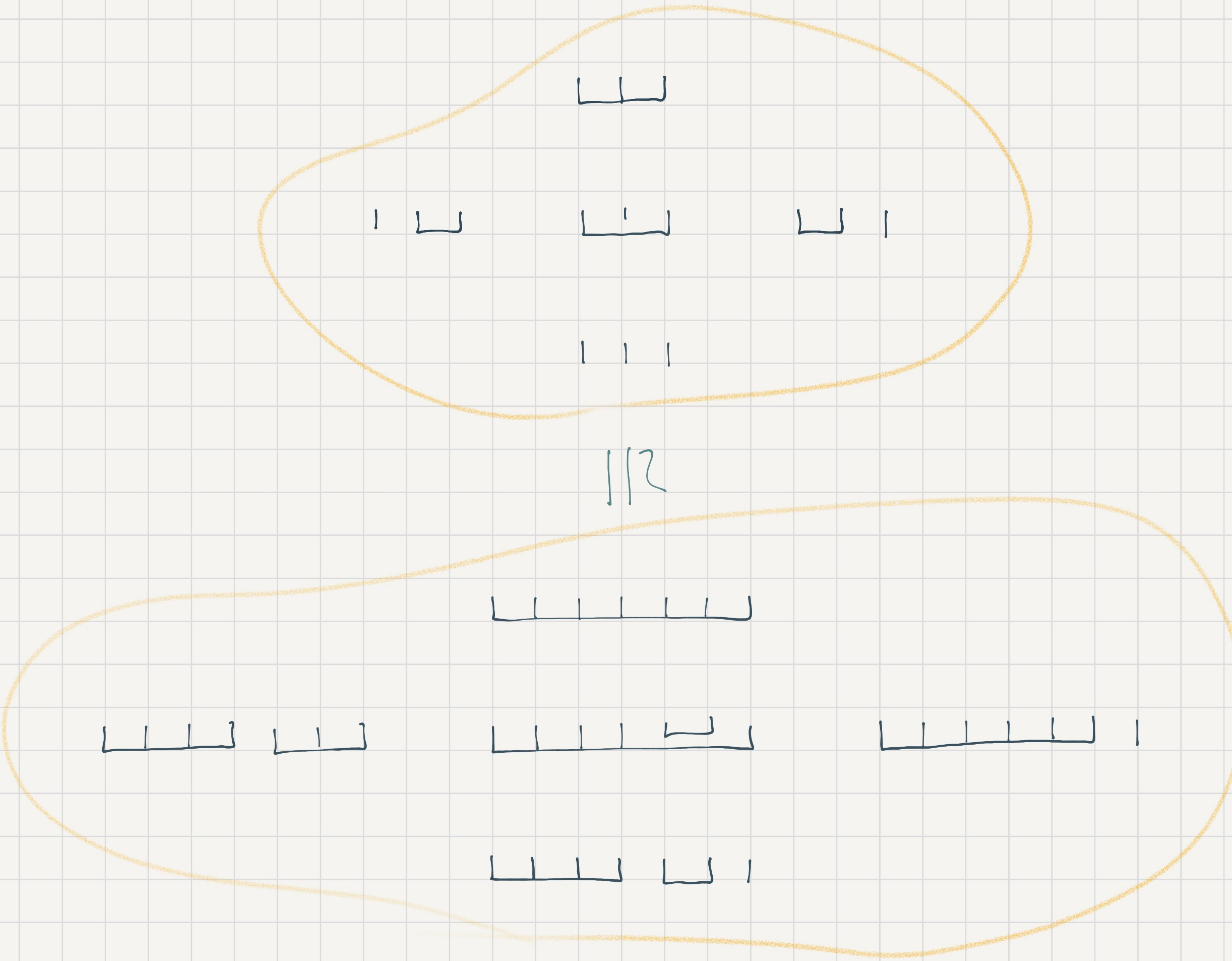
$$i(k) \sim_{\hat{\tau}} i(l) \iff k \sim_\tau l$$

Non-crossing indeed

Example.

$\mathring{i}(1) = 4 \qquad \mathring{i}(2) = 6 \qquad \mathring{i}(3) = 7$





Properties of $\hat{\ }$.

* $\hat{\ }$ is injective

* $\hat{1}_m = 1_n$

* $\hat{0}_m = \{\{1,\dots,i(1)\},\{i(1)+1,\dots,i(2)\},\dots,\{i(m-1)+1,\dots,i(m)\}\}$

* $\hat{\ }$ is order preserving : $\sigma \underset{NC(m)}{\leq} \pi \implies \hat{\sigma} \underset{NC(n)}{\leq} \hat{\pi}$

* $\operatorname{Im} \hat{\ } = \widehat{NC(m)} = [\hat{0}_m, \hat{1}_m] = [\hat{0}_m, 1_n] \subseteq NC(n)$.

To summarize, $\hat{\ }$ is a lattice isomorphism between $NC(m)$ and $[\hat{0}_m, 1_n]$.

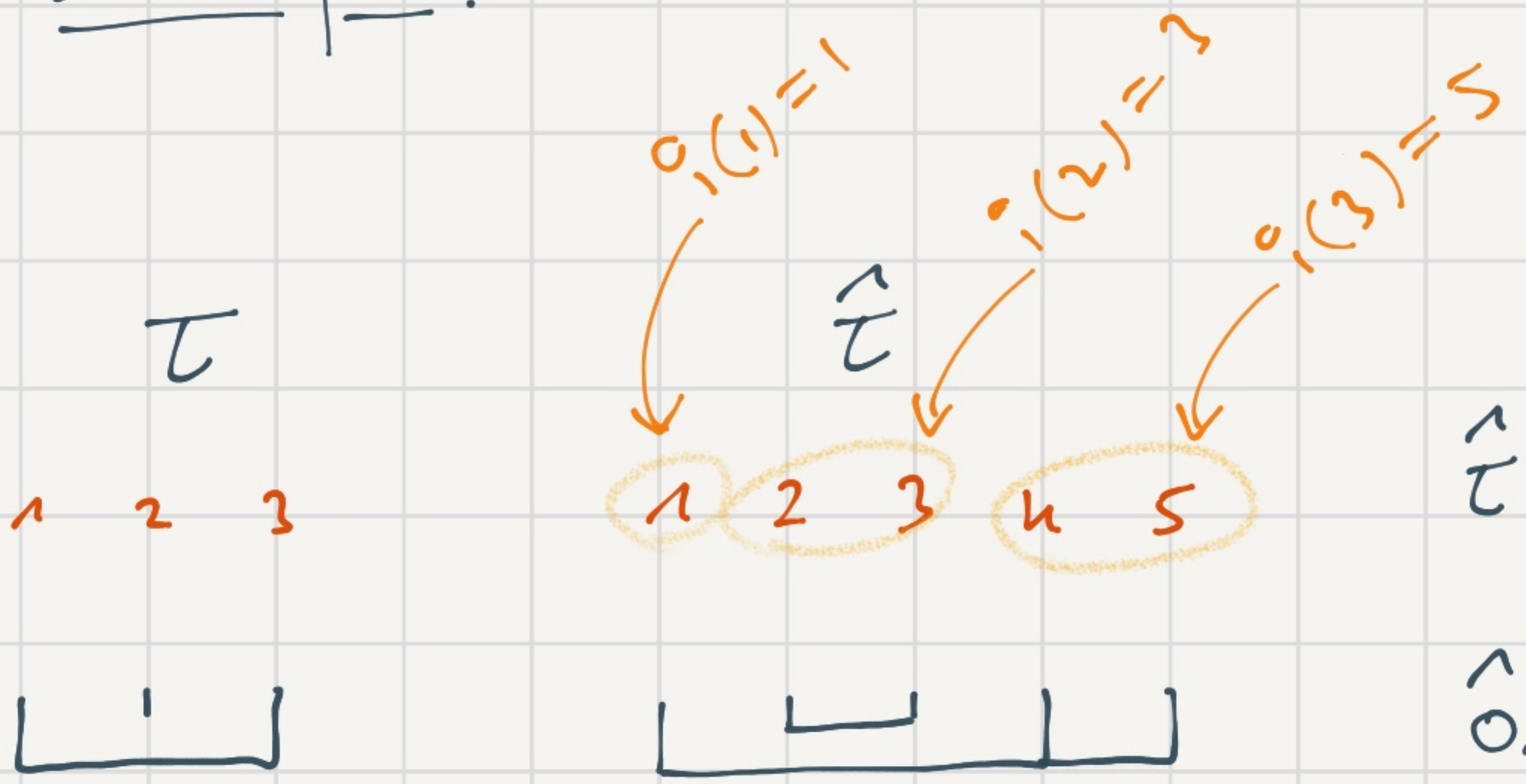
* Moreover, $\forall \sigma \leq \pi \in NC(m)$, $\hat{\ }\big|_{[\sigma,\pi]}$ is a lattice isomorphism to $[\hat{\sigma},\hat{\pi}]$ & $\mu(\sigma,\pi) = \mu(\hat{\sigma},\hat{\pi})$.

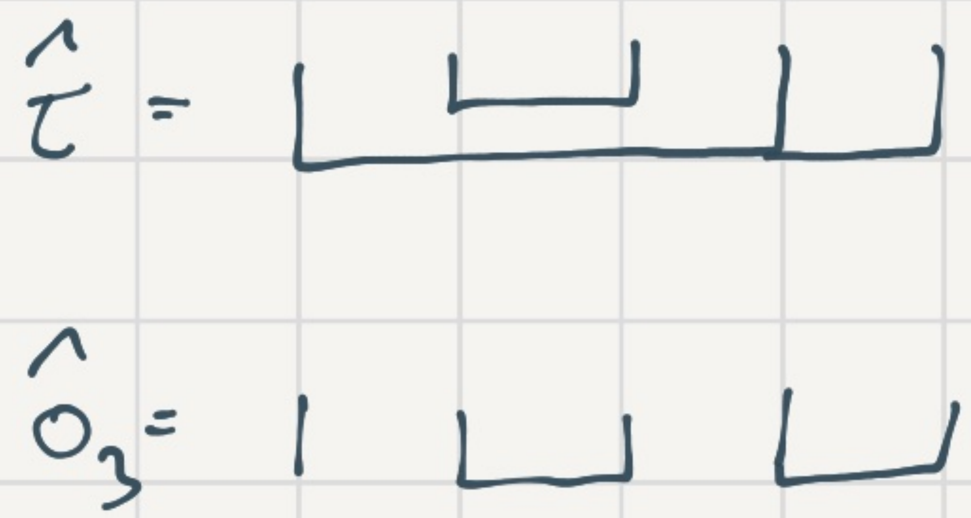
with the above notation

Theorem. $\forall \tau \in NC(m)$

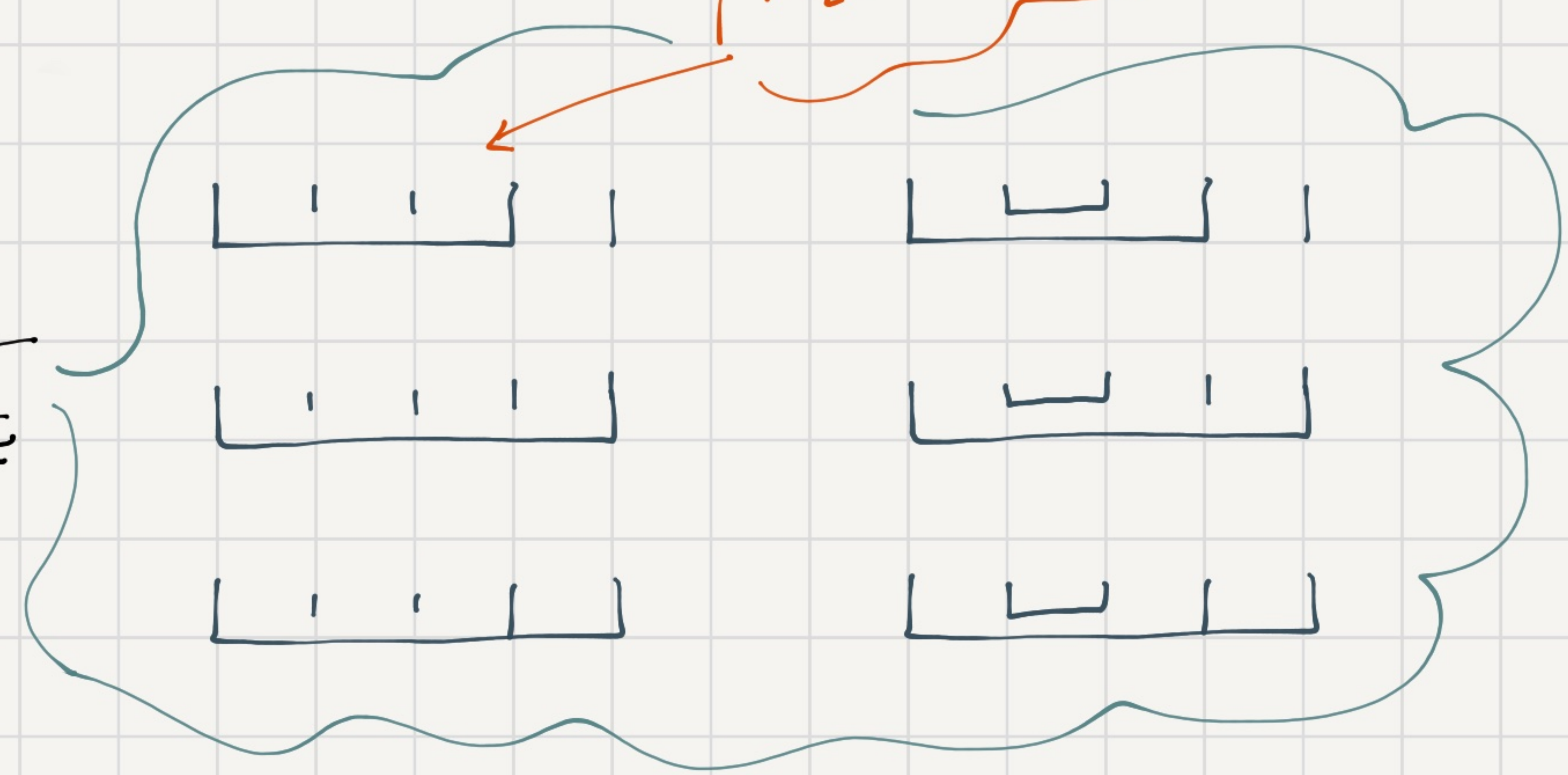
$$\kappa_\tau[a_1 \cdots a_{i(1)}, \dots, a_{i(m-1)+1} \cdots a_{i(m)}] = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \hat{0}_m = \hat{\tau}}} \kappa_\pi[a_1, \dots, a_n].$$

Example.





$$\kappa_{\sqcup\!\sqcup}[a_1, a_2a_3, a_4a_5] = \sum_\pi$$



$\kappa_2(a_1, a_4) \cdot \kappa_1(a_2) \cdot \kappa_1(a_3) \cdot \kappa_1(a_6)$

pf. Denote $A_j = a_{i(j-1)+1} \cdots a_{i(j)}$, $j \in [m]$. Then,

$$K_\tau[A_1,\dots,A_m] = \sum_{\substack{\pi \in NC(m) \\ \pi \le \tau}} \underbrace{\varphi_\pi[A_1,\dots,A_m]}_{\varphi_{\hat\pi}[a_1,\dots,a_n]} \, \underbrace{\mu(\pi,\tau)}_{\mu(\hat\pi,\hat\tau)}$$

By the way we defined ^

Properties above

$$= \sum_{\substack{\pi \in NC(m) \\ \pi \le \tau}} \varphi_{\hat\pi}[a_1,\dots,a_n]\, \mu(\hat\pi,\hat\tau)$$

$[0_m,\tau] = [\hat 0_m, \hat\tau]$

$$= \sum_{\hat 0_m \le \sigma \le \hat\tau} \varphi_\sigma[a_1,\dots,a_n]\, \mu(\sigma,\hat\tau)$$

Partial Mobius Inversion Formula

$$= \sum_{\substack{\pi \in NC(n) \\ \pi \vee \hat 0_m = \hat\tau}} K_\pi[a_1,\dots,a_n]$$

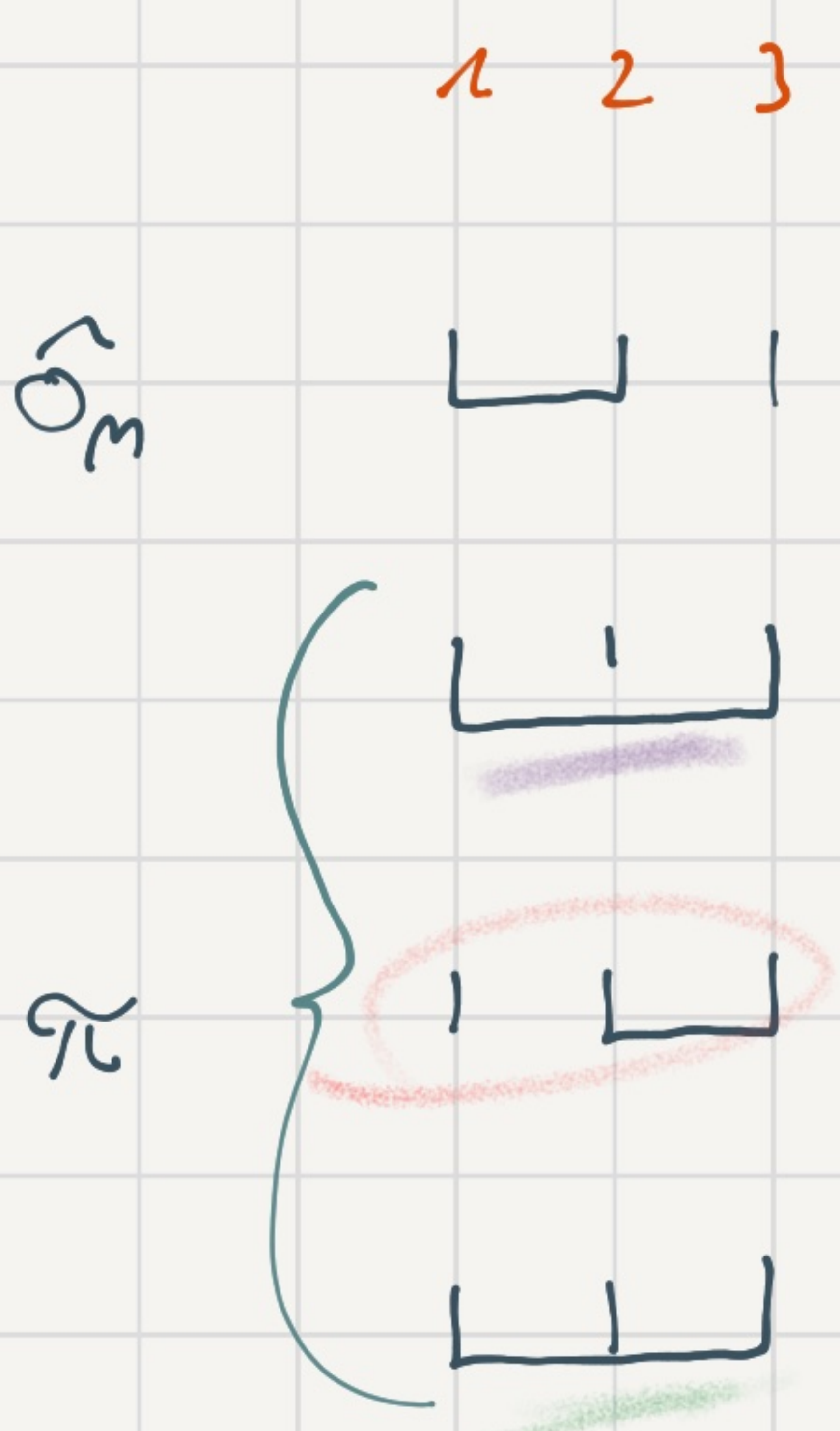
∎

Corollary. with the theorem's notation

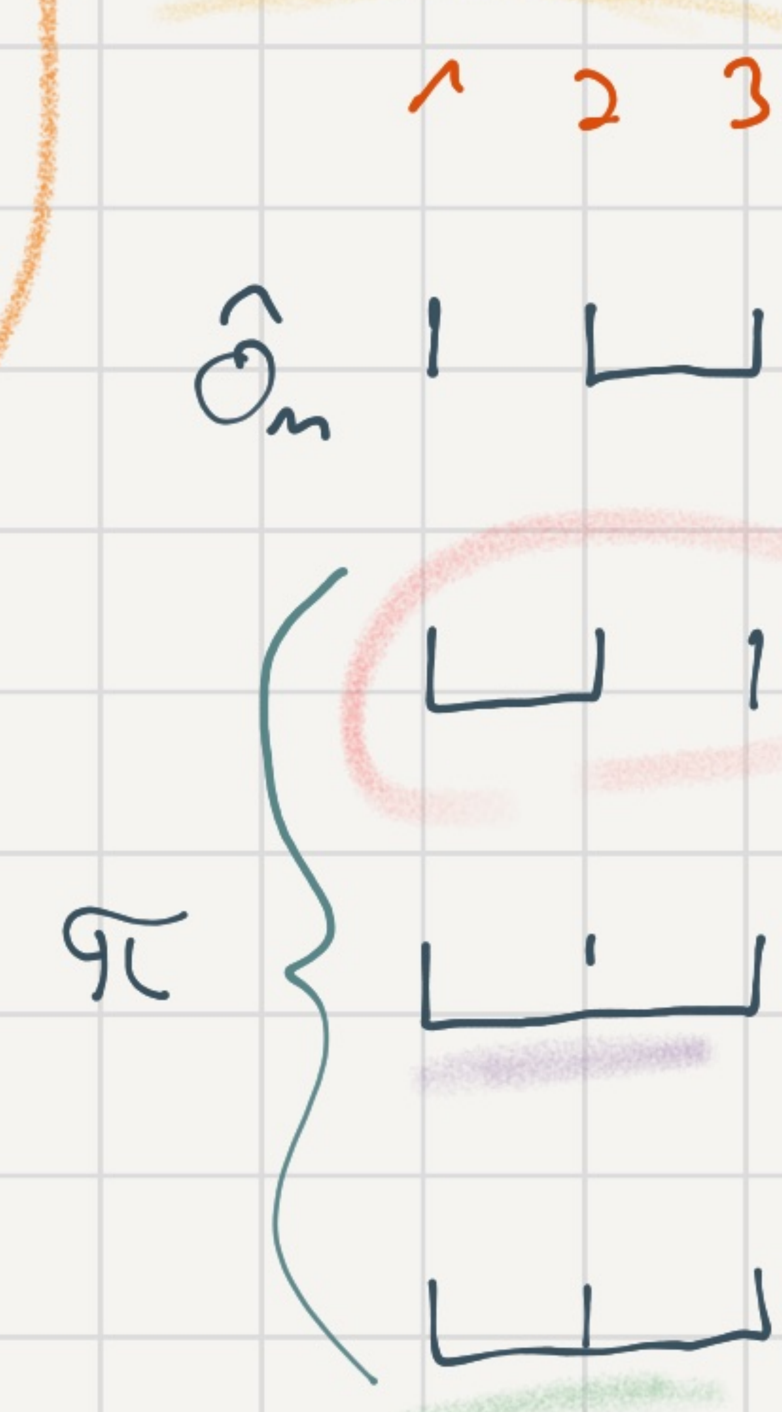
$$K_m(a_1 \cdots a_{i(1)}, \dots, a_{i(m-1)+1} \cdots a_{i(m)}) = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \hat{\sigma}_m = 1_n}} K_\pi[a_1, \dots, a_n]$$

Example

$$K_2(a_1 a_2, a_3) = K_2(a_1, a_3) K_1(a_2) + K_1(a_1) K_2(a_2, a_3) + K_3(a_1, a_2, a_3).$$



$$K_2(a_1, a_2 a_3) = K_2(a_1, a_2) K_1(a_3) + K_2(a_1, a_3) K_1(a_2) + K_3(a_1, a_2, a_3)$$



Free independence

&

free cumulants

Free cumulants are important in FPT because through them, freeness is transparent.

Informal theorem.

Random variables are free $\iff$ their mixed cumulants vanish

As a preliminary step, we prove

Proposition. Let $(\mathcal{A}, \varphi)$ be a ncps with free cumulants $(\kappa_n)_{n\geq 1}$. Let $a_1, \dots, a_n \in \mathcal{A}$, $n \geq 2$, s.t. $\exists i \in [n]$ with $a_i = 1$. Then,

$$\kappa_n(a_1, \dots, a_n) = 0$$

Pf. For simplicity, we prove for the case where $a_n = 1$.

Namely, we prove that $\kappa_n(a_1, \dots, a_{n-1}, 1) = 0$, and proceed by induction.

Base $n=2$. $\kappa_2(a_1, 1) = \varphi(a_1 \cdot 1) - \varphi(a_1)\varphi(1) = 0$

Step $n>2$.

$$\varphi(a_1 \cdots a_{n-1} \cdot 1) = \sum_{\pi \in NC(n)} \kappa_\pi[a_1, \dots, a_{n-1}, 1]$$

$$= \kappa_n(a_1, \dots, a_{n-1}, 1) + \sum_{\substack{\pi \in NC(n) \\ \pi \neq 1_n}} \kappa_\pi[a_1, \dots, a_{n-1}, 1]$$

By induction, for π to contribute to the sum, n must be in its own block. Namely, $\pi = \sigma \cup \{\{n\}\}$ for some $\sigma \in NC(n-1)$.

For such π,

$$\kappa_\pi[a_1,\dots,a_{n-1},1] = \kappa_\sigma[a_1,\dots,a_{n-1}]\,\overbrace{\kappa_n(1)}^{1}$$

Thus,

$$\varphi(a_1\cdots a_{n-1}\cdot 1) = \kappa_n(a_1,\dots,a_{n-1},1) + \underbrace{\sum_{\sigma\in NC(n-1)} \kappa_\sigma[a_1,\dots,a_{n-1}]}_{\varphi(a_1\cdots a_{n-1})}$$

$$\Rightarrow \quad \kappa_n(a_1,\dots,a_{n-1},1) = 0.$$

∎

We turn to prove a central result in FPT.

Theorem (Vanishing of mixed cumulants).

Let $(\mathcal{A}, \varphi)$ be a ncps with free cumulants (κ_n).

Let $(\mathcal{A}_i)_{i \in I}$ be unital subalgebras of $\mathcal{A}$. Then the following are equivalent:

(F) $(\mathcal{A}_i)_{i \in I}$ are freely independent.

(K) $\forall n \geq 2 \quad \forall i(1), \dots, i(n) \in I \quad \forall a_j \in \mathcal{A}_{i(j)}$

$$\exists l, k \in [n] \text{ s.t. } i(l) \neq i(k) \implies \kappa_n(a_1, \dots, a_n) = 0$$

$K \Rightarrow F$. We're given $a_j \in A_{i(j)}$, $j \in [n]$ with $\varphi(a_j) = 0$ & $i(1) \neq i(2) \neq \cdots \neq i(n)$, and wish to show that $\varphi(a_1 \cdots a_n) = 0$.

Well,

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi [a_1, \dots, a_n]$$

Fix π. By non-crossing π must contain an interval $I = \{k, k+1, \dots, k+r\}$

and so

$$\kappa_\pi [a_1, \dots, a_n] = \boxed{\quad} \cdot \kappa_{r+1}(a_k, a_{k+1}, \dots, a_{k+r}) = \begin{cases} r = 0: & \kappa_1(a_k) = \varphi(a_k) \overset{\text{centered}}{=} 0 \\ r > 0: & \kappa_r(\underbrace{a_k, a_{k+1}}_{i(k) \neq i(k+1)}, \dots) \underset{\text{assumption}}{=} 0 \end{cases}$$

$F \Rightarrow K.$ If $\varphi(a_j) = 0 \quad \forall j$ & $i(1) \neq i(2) \neq \cdots \neq i(n)$ then

$$\kappa_n(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} \varphi_\pi[a_1, \dots, a_n]\, \mu(\pi, 1_n) = 0.$$

π contains an interval block : $\dots \varphi_r(a_k \cdots a_{k+r})$

$= 0$

centered / freeness
$(r=0)$ / $(r>0)$

Getting rid of the centeredness assumption:

centered case

$$0 = \kappa_n(a_1 - \varphi(a_1)\cdot 1, \dots, a_n - \varphi(a_n)\cdot 1) = \kappa_n(a_1, \dots, a_n)$$

when expanding, every summand containing $\varphi(a_j)\cdot 1$ vanishes by the previous proposition

Recap. We proved that $\forall n \geq 2$ $\forall\, i(1) \neq i(2) \neq \cdots \neq i(n)$,

$\kappa_n(a_1, \ldots, a_n) = 0$ assuming (F).

With this we proceed by induction on n.

Base $n=2$. a_1, a_2 are free then $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ and

so $\kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2) = 0$.

Step. Let $k, \ell \in [n]$ be s.t. $i(k) \neq i(\ell)$. We wish to prove that $\kappa_n(a_1 \cdots a_n) = 0$ (under (F)). If $i(1) \neq i(2) \neq \cdots \neq i(n)$ we're done. Otherwise, multiply neighboring terms from the same algebra, i.e. $a_1 \cdots a_n = A_1 \cdots A_m$

$i(k) \neq i(\ell)$ not alternating

where $2 \leq m < n$, where A_i & A_{i+1} come from different algebras.

Thus, by our previous theorem, with $\sigma = \hat{0}_m$ capturing the block structure

$$0 = \kappa_m[A_1, \dots, A_m] = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \sigma = 1_n}} \kappa_\pi[a_1, \dots, a_n]$$

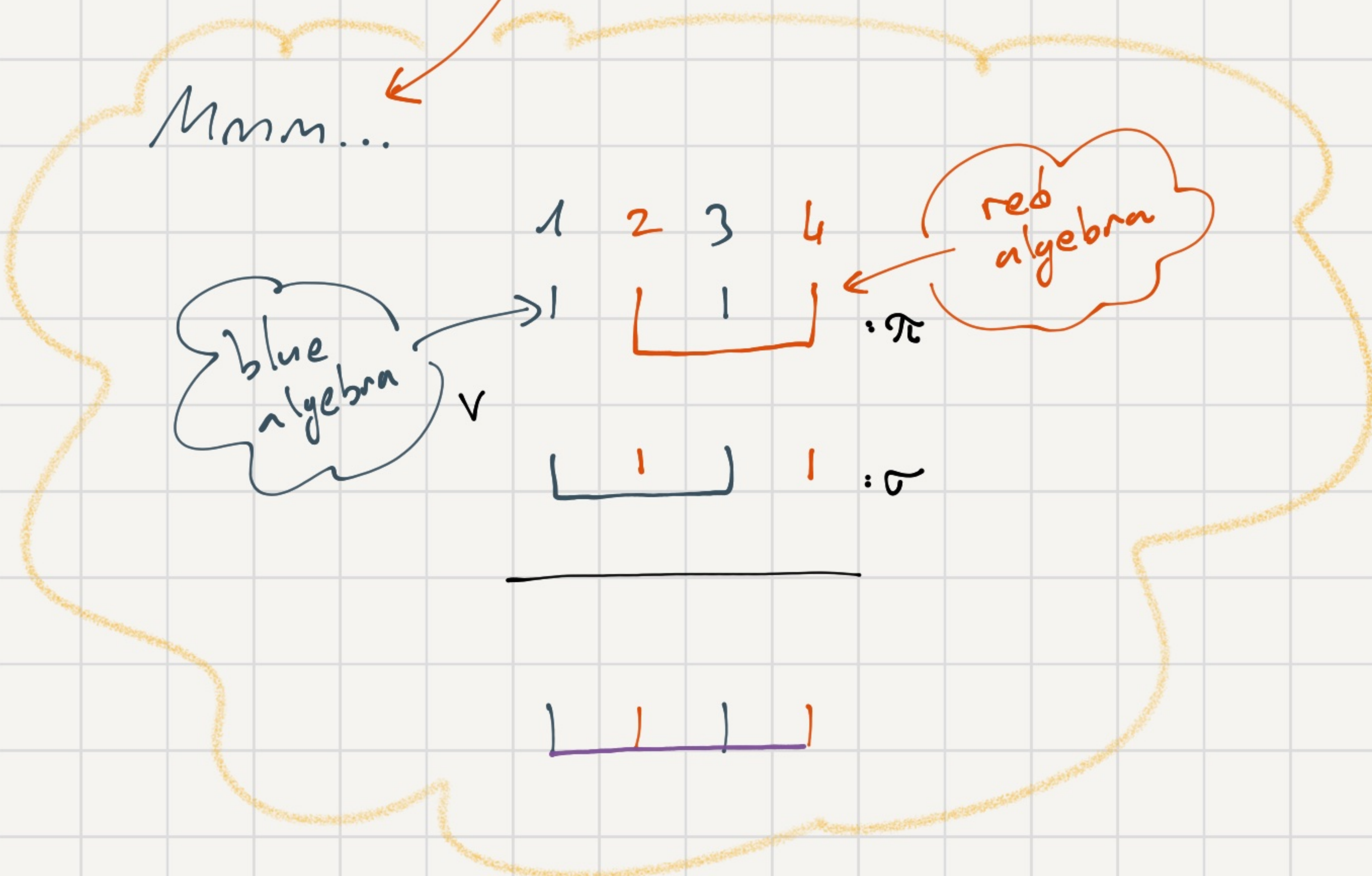
alternating

$$= \kappa_n(a_1, \dots, a_n) + \sum_{\substack{\pi \in NC(n) \\ \pi \vee \sigma = 1_n \\ \pi \neq 1_n}} \kappa_\pi[a_1, \dots, a_n]$$

Fix $1_n \neq \pi \in NC(n)$ s.t. $\pi \vee \sigma = 1_n$. Then $\kappa_\pi[a_1, \dots, a_n] = \prod_{V \in \pi} \kappa(V)[a_1, \dots, a_n]$

As $\pi \neq 1_n$ each $\kappa(V)[a_1, \dots, a_n] = \kappa_s[a_{i_1}, \dots, a_{i_s}]$ for $s = |V| < n$ and so induction applies. Thus, for π to contribute, each block of π must contain elements from the same algebra.

So both σ & π have blocks containg elements from the same algebra, hence so is $\sigma \vee \pi$ in contradiction to $\sigma \vee \pi = 1_n$.



This is actually true due to the fact that σ is an "interval partition": $\sigma = \{\{1,\dots,i(1)\}, \{i(1)+1,\dots,i(2)\},\dots\}$. In such case $\sigma \vee_{NC(n)} \pi = \sigma \vee_{P(n)} \pi$ as you proved in the problem set. ∎