Valuation Rings

Unit 5

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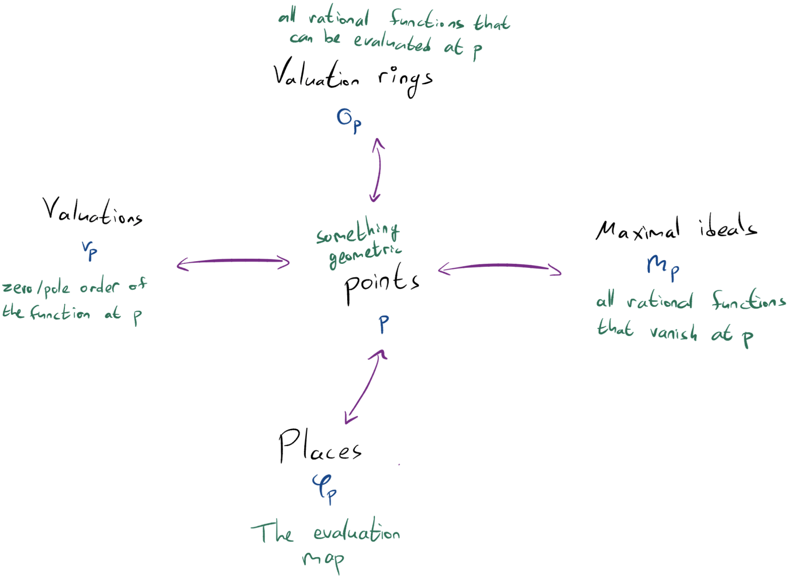
Recall



all rational functions that can be evaluated at p
Valuation rings
$\mathcal{O}_p$
Valuations
v_p
zero/pole order of the function at p
something geometric
points
p
Maximal ideals
m_p
all rational functions that vanish at p
Places
φ_p
The evaluation map

Overview

Valuation rings

Definition 1 (Valuation rings)

Let R be a domain with fraction field F. We say that R is a valuation ring if for all $a \in F^\times$ either $a \in R$ or $a^{-1} \in R$ (or both).

We turn to show that a valuation gives rise to a valuation ring.

Definition 2

For a valuation v on a field F, define

$$\mathcal{O}_v = \{a \in F \mid v(a) \geq 0\}.$$

From valuations to valuation rings

Claim 3

Let v be a valuation on a field F. Then, $\mathcal{O}_v$ is a valuation ring with $\operatorname{Frac}\mathcal{O}_v = \mathsf{F}$. Moreover,

$$\mathcal{O}_v^\times = \{a \in \mathsf{F} \mid v(a) = 0\}.$$

Proof.

We first show $\mathcal{O}_v$ is a ring.

- Note that $0 \in \mathcal{O}_v$ as $v(0) = \infty \geq 0$.
- As $v(a+b) \geq \min(v(a), v(b))$, $\mathcal{O}_v$ is closed under addition.
- Similarly for multiplication since $v(ab) = v(a) + v(b)$.

The associativity and the distributive law are induced from F and so $\mathcal{O}_v$ is a ring.

Take $a \in \mathsf{F}^\times$. Recall that $v(a^{-1}) = -v(a)$ and so one of $v(a)$, $v(a^{-1}) \geq 0$, namely, one of $a, a^{-1} \in \mathcal{O}_v$. Thus, $\mathcal{O}_v$ is a valuation ring.

From valuations to valuation rings

Proof.

Clearly, $\operatorname{Frac} \mathcal{O}_v \subseteq F$ (as usual, up to isomorphism). For the other inclusion, take $a \in F$. If $a \in \mathcal{O}_v$ we are done; otherwise, $a^{-1} \in \mathcal{O}_v$ and so $a = \frac{1}{a^{-1}} \in \operatorname{Frac} \mathcal{O}_v$.

It remains to show that $a \in \mathcal{O}_v$ is a unit iff $v(a) = 0$. Indeed,

$$\begin{aligned} a \in \mathcal{O}_v^\times \quad &\iff \quad a^{-1} \in \mathcal{O}_v \text{ and } a \in \mathcal{O}_v \\ &\iff \quad v(a^{-1}) \geq 0 \text{ and } v(a) \geq 0 \\ &\iff \quad -v(a) \geq 0 \text{ and } v(a) \geq 0 \\ &\iff \quad v(a) = 0. \end{aligned}$$

□

From valuations to valuation rings

Definition 4

Let v be a valuation on a field F. Then, $\mathcal{O}_v$ is called the valuation ring of v.

Claim 5

Let v, v' be valuations on F. Then, v, v' are equivalent iff $\mathcal{O}_v = \mathcal{O}_{v'}$.

Proof.

The proof readily follows by definition, which we recall here

$$\begin{aligned} a \in \mathcal{O}_v \quad &\iff \quad v(a) \geq 0, \\ a \in \mathcal{O}_{v'} \quad &\iff \quad v'(a) \geq 0. \end{aligned}$$

□

Valuations and valuation rings

To summarize, so far we established

Overview

The maximal ideal of a valuation ring

Claim 6

Let R be a valuation ring with field of fractions F. Then, $\mathfrak{m} = \mathrm{R} \setminus \mathrm{R}^\times$ is the unique maximal ideal of R.

Proof.

Since a maximal ideal cannot contain a unit, if $\mathfrak{m}$ is an ideal then it is maximal and unique. We are thus left to prove that $\mathfrak{m}$ is an ideal.

Take $a \in \mathfrak{m}$, $r \in \mathrm{R}$. Note that $ra \notin \mathrm{R}^\times$ as otherwise

$$a^{-1} = (ra)^{-1} r \in \mathrm{R}$$

and so $a \in \mathrm{R}^\times$ in contradiction to $a \in \mathfrak{m} = \mathrm{R} \setminus \mathrm{R}^\times$. Thus, $ra \in \mathrm{R} \setminus \mathrm{R}^\times = \mathfrak{m}$.

The maximal ideal of a valuation ring

Proof.

It is left to show that $\mathfrak{m}$ is closed under addition.

Take $a, b \in \mathfrak{m} \setminus \{0\}$. Since R is a valuation ring, we may assume wlog that $\frac{a}{b} \in R$. Thus, $1 + \frac{a}{b} \in R$ and so

$$a + b = b\left(1 + \frac{a}{b}\right) \in \mathfrak{m}.$$

□

The maximal ideal of a valuation ring

Since a valuation ring has a unique maximal ideal, the former determines the latter.

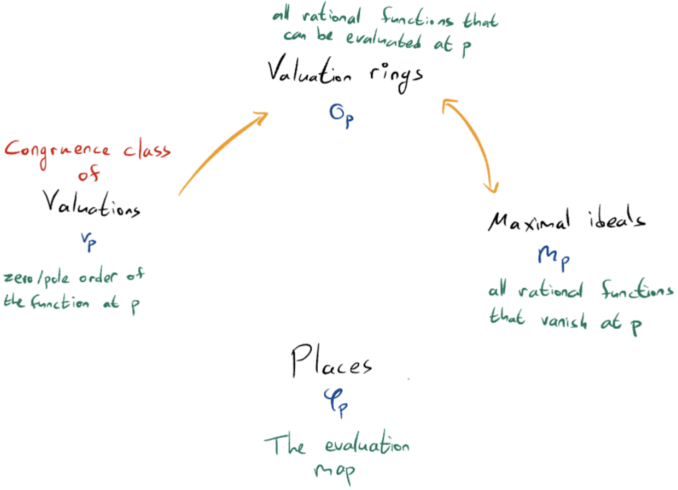
In the other direction, if $\mathfrak{m}$ is a maximal ideal of some valuation ring R whose fraction field is F then that valuation ring is determined by $\mathfrak{m}$. Indeed, I leave it for you as an exercise to prove that

$$\mathsf{R} = \left\{ a \in \mathsf{F}^{\times} \;\middle|\; \frac{1}{a} \notin \mathfrak{m} \right\} \cup \{0\}.$$

Intuitively, *a* is defined at a point exactly when its inverse does not vanish.

The maximal ideal of a valuation ring

To summarize, so far we established



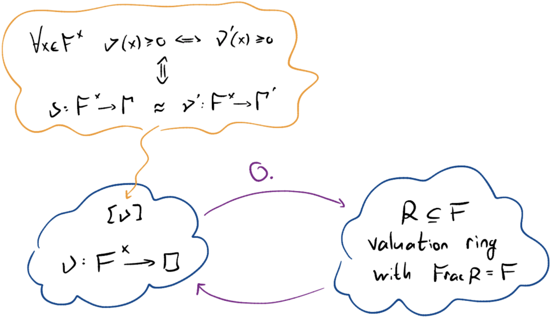
Overview

From valuation rings to valuations

Recall Claim 5 which stated that for equivalent valuations v, v' we have $\mathcal{O}_v = \mathcal{O}_{v'}$, and only for those.

Theorem 7

Let F *be a field. The map* $v \mapsto \mathcal{O}_v$ *induces a one to one map between congruence class of valuations on* F *and valuation rings with fraction field* F*.*



To prove Theorem 7 we recall a claim from Unit 3.

Claim 8

Let Γ be an abelian group with a submonoid Γ_+ satisfying

$$\Gamma_+ \cap (-\Gamma_+) = \{0\},$$
$$\Gamma_+ \cup (-\Gamma_+) = \Gamma.$$

Define an order on Γ by

$$\alpha \leq \beta \quad \Longleftrightarrow \quad \beta - \alpha \in \Gamma_+.$$

Under this order, Γ is an ordered group.

In fact, we will invoke the claim in its multiplicative form.

Claim 9

Let Γ be an abelian group with a submonoid Γ_+ satisfying

$$\Gamma_+ \cap (\Gamma_+)^{-1} = \{1\},$$
$$\Gamma_+ \cup (\Gamma_+)^{-1} = \Gamma.$$

Define an order on Γ by

$$\alpha \leq \beta \iff \beta\alpha^{-1} \in \Gamma_+.$$

Under this order, Γ is an ordered group.

Proof. (of Theorem 7)

Given Claim 5, to prove the theorem, we define an inverse map

$$\mathrm{R} \mapsto (v : \mathrm{F}^\times \to \Gamma).$$

Let R be a valuation ring with $\operatorname{Frac}\mathrm{R} = \mathrm{F}$. The trouble is that we somehow need to come up with an ordered group Γ.

Denote $\mathrm{S} = \mathrm{R} \setminus \{0\}$. Note that S, S^{-1} are submonoids of the multiplicative group of $\mathrm{F}^\times$, both containing $\mathrm{R}^\times$. Since R is a valuation ring,

$$S \cup S^{-1} = \mathrm{F}^\times,$$
$$S \cap S^{-1} = \mathrm{R}^\times.$$

Proof.

$$S \cup S^{-1} = \mathrm{F}^{\times}, \quad S \cap S^{-1} = \mathrm{R}^{\times}.$$

As $\mathrm{R}^{\times}$ is a submonoid of both monoids S, S^{-1}, we can take congruence classes and get

$$S \big/ \mathrm{R}^{\times} \cup S^{-1} \big/ \mathrm{R}^{\times} = \mathrm{F}^{\times} \big/ \mathrm{R}^{\times}$$
$$S \big/ \mathrm{R}^{\times} \cap S^{-1} \big/ \mathrm{R}^{\times} = \mathrm{R}^{\times} \big/ \mathrm{R}^{\times} = \{1\}$$

Thus, by Claim 9, $\mathrm{F}^{\times} \big/ \mathrm{R}^{\times}$ is an ordered group.

Proof.

Consider the projection map

$$w : \mathsf{F}^\times \to \mathsf{F}^\times \big/ \mathsf{R}^\times .$$

By Claim 9, we have that

$$\begin{aligned} w(a) \le w(b) \quad &\iff \quad \frac{w(b)}{w(a)} \in \mathsf{S} \big/ \mathsf{R}^\times \\ &\iff \quad w(ba^{-1}) \in \mathsf{S} \big/ \mathsf{R}^\times \\ &\iff \quad ba^{-1} \in S. \end{aligned}$$

Valuations and valuation rings

Proof.

With this we turn to prove that

$$w : F^{\times} \to F^{\times} / R^{\times}$$

is a valuation.

Take $a, b \in F^{\times}$ with $a + b \neq 0$. Assume wlog $w(a) \leq w(b)$. Then,

$$\frac{a+b}{a} = 1 + \frac{b}{a} \in S$$

and so $w(a+b) \geq w(a)$.

Since w is a group homomorphism, $w(ab) = w(a)w(b)$. Thus, w is a valuation on F.

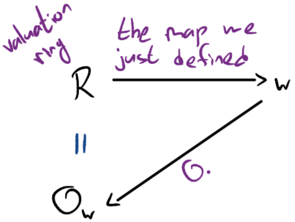
Valuations and valuation rings

Proof.

Note that the map $R \mapsto w$ that we just defined is inverse to the map $v \mapsto \mathcal{O}_v$. On the one hand,

$$\mathcal{O}_w = \{a \in \mathsf{F} \mid w(a) \geq 1\} = \{a \in \mathsf{F} \mid a \in R\} = R.$$

□



Valuations and valuation rings

Proof.

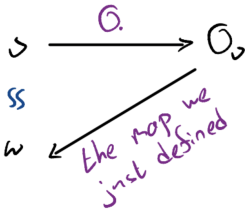
On the other hand, if we start with v, then

$$v(a) \geq 0 \quad \Longleftrightarrow \quad a \in \mathcal{O}_v.$$

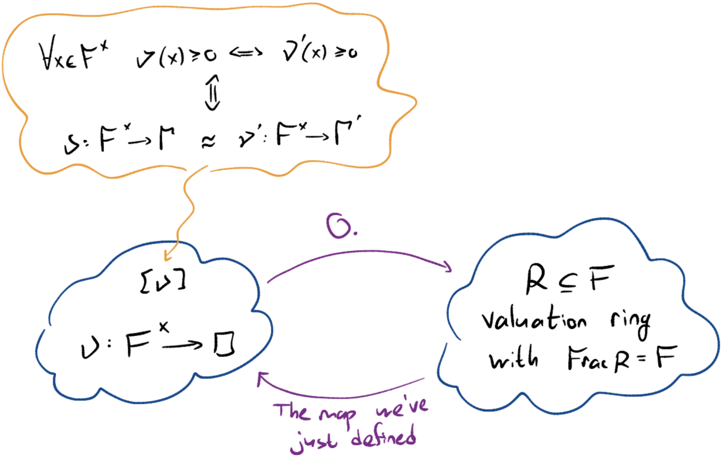
By the definition of the map $\mathcal{O}_v \mapsto w$ (in additive notation)

$$a \in \mathcal{O}_v \quad \Longleftrightarrow \quad w(a) \geq 0.$$

Thus, w is equivalent to v. □



Valuations and valuation rings



Overview

Example

Recall the example from the previous unit. Let $\mathsf{K} = \mathbb{F}_q$, let

$$f(x,y) = y^2 - x^3 + x \in K[x,y],$$

and consider the domain

$$C_f = \mathsf{K}[x,y] \Big/ \langle f(x,y) \rangle$$

whose field of fractions is denoted by $\mathsf{K}_f = \mathrm{Frac}\, C_f$.

Consider the point $\mathfrak{o} = (0,0)$ on the curve. We proved that

$$\upsilon_{\mathfrak{o}}(A(x) + B(x)y) = \min\left(\upsilon_{\mathfrak{o}}(A(x)), 1 + \upsilon_{\mathfrak{o}}(B(x))\right),$$

where $A(x), B(x) \in \mathsf{K}(x)$.

Example

$$v_{\circ}(A(x) + B(x)y) = \min\left(v_{\circ}(A(x)), 1 + v_{\circ}(B(x))\right).$$

Let $\mathcal{O}_{\circ}$ be the valuation ring corresponding to $v_{\circ}$, namely,

$$\mathcal{O}_{\circ} = \{z \in \mathsf{K}_f \mid v_{\circ}(z) \geq 0\}.$$

Therefore,

$$\mathcal{O}_{\circ} = \left\{ \frac{a(x)}{b(x)} + y\frac{c(x)}{d(x)} \;\middle|\; b(0), d(0) \neq 0 \right\},$$

with the understanding that $a(x), b(x)$ are coprime and so are $c(x), d(x)$.

Exercise. What is $\mathfrak{m}_{\circ}$?

To summarize