Hilbert's Basis Theorem and Algebras

Gil Cohen

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Recall

Lemma

Let $A \subseteq B$ be rings s.t.

- *A is a noetherian ring, and*
- *B is a f.g. A-module.*

Then, B is a noetherian ring.

Discussion

Let $f(x, y) \in \bar{K}[x, y]$. Write it as

$$f(x, y) = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_0(x) \in \bar{K}[x, y].$$

Recall that $A = \bar{K}[x]$ is noetheiran (even PID etc). Further, if $a_n(x) = 1$ then the ring $B = C_f$ is a f.g. A-module.

Discussion

In particular,

$$B = A + Ay + Ay^2 + \cdots + Ay^{n-1}$$

or, more precisely,

$$B = A(1 + \langle f \rangle) + A(y + \langle f \rangle) + \cdots + A(y^{n-1} + \langle f \rangle).$$

The above lemma then shows that C_f is a noetherian ring.

In this unit we are going to strengthen the lemma so to conclude that C_f is a noetherian ring even without assuming anything about $a_n(x)$.

Theorem (Hilbert's Basis Theorem)

A ring A is noetherian $\iff$ $A[y]$ is noetherian.

Proof.

In the recitation. □

Corollary

Let $A \subseteq B$ be rings s.t:

- *A is a noetherian ring, and*
- *$\exists b_1, \ldots, b_n \in B$ s.t. $B = A[b_1, \ldots, b_n]$.*

Then, B is a noetherian ring.

Proof.

Let $F = A[x_1, \ldots, x_n]$ be the ring of polynomials in n variables over A. By Hilbert's basis theorem, F is a noetherian ring. Consider the ring homomorphism

$$\varphi : F \to B$$
$$f(x_1, \ldots, x_n) \mapsto f(b_1, \ldots, b_n)$$

Let $K = \ker(\varphi)$. Since φ is surjective, $B \cong F/K$. The proof follows then since F is noetherian and since quotient of a noetherian ring is noetherian. $\square$

Corollary

Let $f(x, y) \in \bar{K}[x, y]$ irreducible. Then, C_f is a noetherian domain.

Proof.

f irreducible $\implies$ C_f is a domain. Assume w.l.o.g that $\deg_y(f) > 0$. Then,

- $A = \bar{K}[x]$ is noetherian, and
- $A \subseteq C_f$. More precisely, $A \hookrightarrow C_f$.
- $C_f = A[y]$. More precisely, $C_f = (A + \langle f \rangle)[y + \langle f \rangle]$.

The proof then follows by the previous corollary. □

Recall (proved in the homework assignment)

Claim

Let $f \in \bar{K}[x,y]$ irreducible. Then, $\dim(C_f) = 1$.

We are now ready to prove a fundamental result connecting yet again algebra and geometry.

Theorem

Let $f \in \bar{K}[x,y]$ irreducible. Then,

$$C_f \text{ Dedekind domain} \iff Z_f(\bar{K}) \text{ is nonsingular}$$

Proof.

By the above, since f is irreducible, C_f is a noetherian domain of dimension 1. It suffices to prove that

$$C_f \text{ integrally closed} \iff Z_f(\bar{K}) \text{ is nonsingular}$$

In previous units we proved:

- Integrally closed is a local property.
- Hilbert's Nullstellensatz: Every maximal ideal of C_f is of the form $M = \langle x - a, y - b \rangle$ for some $(a, b) \in Z_f(\bar{K})$.
- $(a, b) \in Z_f(\bar{K})$ is nonsingular $\iff (C_f)_M$ is a PID.
- Since $(C_f)_M$ is a local noetherian domain of dimension 1, $(C_f)_M$ is a PID $\iff (C_f)_M$ is integrally closed.

□

Definition

Let A be a commutative ring. A ring M is an *A-algebra* if there exists a ring homomorphism $\phi : A \to M$ s.t. the elements $\phi(A)$ commute with all elements of M.

Remark

If M is an A-algebra then M is in particular an A-module. Indeed, one can define

$$\mu : A \times M \to M$$
$$(a, m) \mapsto \phi(a)m$$

Example

- $K[x_1, \ldots, x_n]$ is a K-algebra.
- If I ideal of $K[x_1, \ldots, x_n]$ then $K[x_1, \ldots, x_n]/I$ is a K-algebra.
- In particular, $C_f = K[x, y]/\langle f(x, y) \rangle$ is a K-algebra.
- The ring of $n \times n$ matrices over a ring A is an A-algebra.

Definition

Let M be an A-algebra. M is said to be a finitely generated A-algebra if $\exists m_1, \ldots, m_n \in M$ s.t. $M = A[m_1, \ldots, m_n]$.

Example

- All the above examples.

Claim

Let M be an A-algebra. M is f.g. A-algebra $\iff$ there exists an integer n and an ideal I of $A[x_1, \ldots, x_n]$ s.t.

$$M \cong A[x_1, \ldots, x_n]/I.$$

Proof.

Clearly, $A[x_1, \ldots, x_n]/I$ is a f.g. A-algebra (with generators $x_1 + I, \ldots, x_n + I$).

M is a f.g. A-algebra $\implies$ $M = A[m_1, \ldots, m_n]$ for $m_1, \ldots, m_n \in M$. Consider the ring homomorphism $\phi : A[x_1, \ldots, x_n] \to M$ that maps $x_i \mapsto m_i$ and fixes A. Let $K = \ker(\phi)$. Note that ϕ is surjective and so $M = A[x_1, \ldots, x_n]/K$. □

Recall

Lemma

Let $A \subseteq B$ be rings such that:

- *A is a noetherian ring, and*
- *B is a f.g. A-module.*

Then, B is a noetherian ring.

The corollary of Hilbert's basis theorem can now be stated as

Lemma

Let $A \subseteq B$ be rings such that:

- *A is a noetherian ring, and*
- *B is a f.g. A-algebra.*

Then, B is a noetherian ring.