Noetherian Rings

Noetherian rings are important and ubiquitous family of rings having a certain finitness condition that guarantees one cannot "factor for ever".

Definition

A ring is noetherian if every ideal in A is f.g.

Remarks & Examples

* One can view a ring A as an A-module. As such the submodules are precisely the ideals in A. So, a ring A is noetherian $\iff$ it is a noetherian A-module.
* PIDs are noetherian
* A noetherian, I ideal in A then A/I noetherian.

Proposition

A noetherian. M f.g A-module. Then, M is notherian.

Proof

Let $M = m_1 A + \cdots + m_n A$. The sequence

$$0 \longrightarrow \operatorname{Ker} f \hookrightarrow \bigoplus_{i=1}^{n} A e_i \xrightarrow{f} M \longrightarrow 0$$

note that

with $f : \bigoplus_{i=1}^{n} A e_i \longrightarrow M$, $e_i \longmapsto m_i$ is exact.

A noetherian $\Longrightarrow \bigoplus A e_i$ notherian

proved before

$\Longrightarrow M$ noetherian

Corollary

$A \subseteq B$ rings.

$$\begin{cases} A \text{ noetherian} \\ \quad \& \\ B \text{ f.g. } A\text{-module} \end{cases} \implies B \text{ noetherian ring.}$$

Proof

An ideal I of B is an A-submodule of the A-module B (I subgroup + $AI \subseteq I$).

B f.g A-module and A notherian $\implies$ I f.g. as an A-module

Previous Proposition

$\implies$ I f.g as an ideal of B

$\boxtimes$

Remark

The assumption that B is a f.g. A-module can be relaxed to f.g A-algebra.

This is the content of Hilbert's basis theorem that we'll prove later on.

Proposition

A integrally closed domain. $K = \operatorname{Frac} A$. L/K separable of degree n. Let B be the integral closure of A in L. Let $\{e_1, \dots, e_n\} \subseteq B$ be a basis for L as a K-vector space.

Then, $\exists\, d \in A$ s.t.

$$e_1 A_1 \oplus \cdots \oplus e_n A \subseteq B \subseteq \frac{e_1}{d} A \oplus \cdots \oplus \frac{e_n}{d} A \subseteq L$$

Proof

First, we proved a basis as above (which is implicitly assumed to exist) always exists. Since L/K finite & separable $L = K(\alpha)$ for some $\alpha \in L$. We may assume $\alpha \in B$ by "clearing denominator". So, we may take $e_i = \alpha^{i-1}$.

Let $\beta \in B$. β induces a unique sequence $x_0, \dots, x_{n-1} \in K$ s.t. $\beta = \sum_{j=0}^{n-1} x_j \alpha^j$. Define M $n \times n$ matrix with $M_{ij} = \sigma_i(\alpha^{j-1})$. Note $M_{ij} \in B$. Indeed, $\alpha \in B \Rightarrow \alpha^{j-1} \in B$

$\Rightarrow \exists\, f(x) \in A[x]$ monic with $f(\alpha^{j-1}) = 0 \Rightarrow f(\sigma_i(\alpha^{j-1})) \underset{\sigma_i|_A = \mathrm{id}_A}{=} \sigma_i(f(\alpha^{j-1})) = \sigma_i(0) = 0 \Rightarrow \sigma_i(\alpha^{j-1}) \in B$.

Now,

$$\begin{pmatrix}\sigma_1(\beta)\\ \vdots \\ \sigma_n(\beta)\end{pmatrix} = M\begin{pmatrix}x_0\\ \vdots \\ x_{n-1}\end{pmatrix} \implies M^*\begin{pmatrix}\sigma_1(\beta)\\ \vdots \\ \sigma_n(\beta)\end{pmatrix} = \begin{pmatrix}\det(M)\cdot x_0\\ \vdots \\ \det(M)\cdot x_{n-1}\end{pmatrix}$$

adjoint matrix to M

M^*'s entries $\in B$ being minors of M. $\sigma_i(\beta)\in B$ arguing as before as $\beta\in B$.

$\implies \det(M)x_i \in B \qquad i=0,\dots,n-1.$

Had it been that $\det(M)x_i\in K$ we could have conclude $\det(M)x_i\in K\cap B = A$ and take $d=\det(M)$. This is not true however. We'll show that $d=\det(M)^2$ is a good choice:

Each $\sigma_i\in G_{L/K}$ can be extended to an automorphism $\eta_i\in G_K$ (that is $\eta_i|_L=\sigma_i$).

Take any automorphism $\eta\in G_K$. $\exists i$ s.t $\eta=\eta_i$. Applying η_i to M permutes its columns $\implies \eta_i(\det M)=\det(\eta_i(M)) = \pm\det M$

$\implies d=(\det M)^2$ is invariant under all automorphisms $\implies d\in K$.

d is also in $\mathcal{B}$ (since $d = \det M^2$ and $\det M \in \mathcal{B}$) $\Rightarrow$ $d \in k \cap \mathcal{B} = A$. A i.c.

We conclude that $dx_i \in k$ and $dx_i \in \mathcal{B}$, indeed $dx_i = \underbrace{\det(M)}_{\in \mathcal{B}} \underbrace{\det(M)\, x_i}_{\in \mathcal{B}}$. Thus, $dx_i \in A$.

Lastly, $d \neq 0$ by a calculation we did before and so for any $\beta \in \mathcal{B}$

$$\beta = x_1 + x_2\alpha + \cdots + x_n\alpha^{n-1}$$

$$= \frac{1}{d}\left(x_1 d + x_2 d\alpha + \cdots + x_n d\alpha^{n-1}\right) \in \frac{A \cdot \overset{e_1}{1}}{d} \oplus \frac{A\overset{e_2}{\alpha}}{d} \oplus \cdots + \frac{A\overset{e_n}{\alpha^{n-1}}}{d}$$

$\blacksquare$

Theorem

A noetherian domain, integrally closed in $K = \mathrm{Frac}(A)$. L/K finite separable extension. Then, B - the integral closure of A in L is a f.g. A-module.

In particular, B is a noetherian ring.

Proof

Under the hypothesis of the theorem we showed that B is an A-submodule of a finitely generated A module $\left(B \subseteq \frac{Ae_1}{d} \oplus \cdots \oplus \frac{Ae_n}{d}\right)$. We proved that this implies that B itself is a f.g. A-module & the "in particular" part.

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