A case study

Based on Nica-Speicher, Chapter 2

Example. Let $(\mathcal{A},\varphi)$ be a $*$-ps. Let $a\in\mathcal{A}$ be s.t.

* $a^*a = 1 \neq aa^*$ (non-unitary isometry)
* a generates $\mathcal{A}$ as a $*$-alg. ($\mathcal{A} = \langle a, a^*, 1\rangle$)

Observe that $\mathcal{A} = \mathrm{sp}\{a^m(a^*)^n \mid m,n\geq 0\}$.

Assumption. are linearly ind.

We'll consider the functional $\varphi(a^m(a^*)^n) = \begin{cases} 1 & m=n=0 \\ 0 & \text{o.w.} \end{cases}$

Realizing non-unitary isometries

__Realization / Representation.__ Consider the Hilbert space

Inner product space, complete w.r.t the induced norm

$$\ell^2 = \{ \xi = (a_k)_k \in \mathbb{C}^{\mathbb{N}} \mid \sum |a_k|^2 < \infty \} \quad \text{where} \quad \langle \xi, \eta \rangle = \sum_{k=0}^{\infty} \xi_k \overline{\eta_k}$$

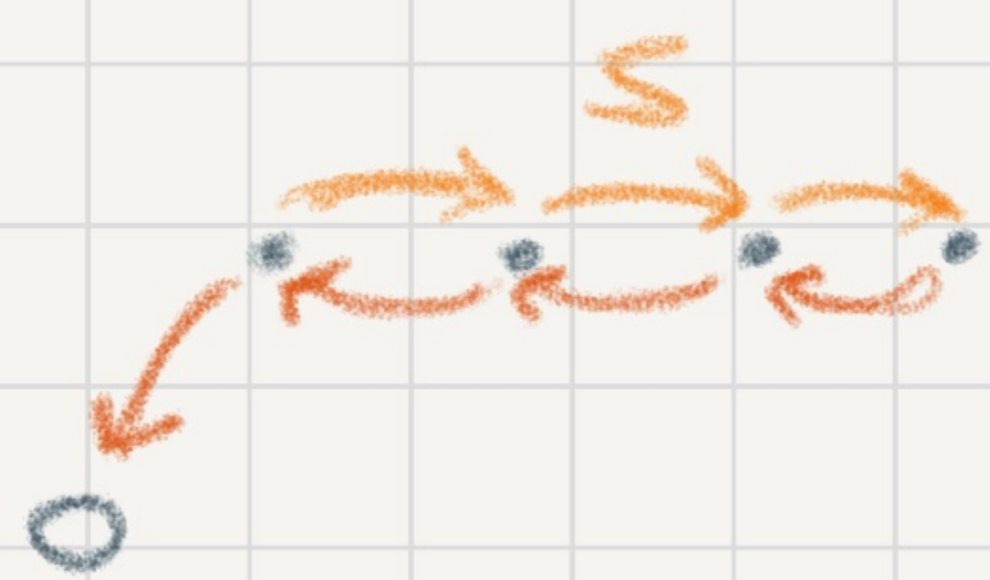
The vectors $(\xi_n)_n$ with $\xi_n = (0, \dots, 0, \overset{n}{1}, 0, \dots)$ form an orthonormal basis for ℓ^2.

Consider the one-sided shift operator S defined by $S\xi_n = \xi_{n+1}$ $n \geq 0$.

Its adjoint S^* is given by $S^*\xi_0 = 0$ & $S^*\xi_n = \xi_{n-1}$ $n \geq 1$.

$\langle S\xi, \eta \rangle = \langle \xi, S^*\eta \rangle$ $\forall \xi, \eta$

why?



Clearly $S^*S = 1_{B(\ell^2)}$ but $SS^* \neq 1$.

It is not hard to prove that $\{ S^m (S^*)^n \mid m, n \geq 0 \}$ are linearly ind.

We can define the linear functional

Hence, a is realized by S

$$\varphi_0(T) = \langle T\xi_0, \xi_0 \rangle \qquad T \in B(\ell^2)$$

linear operators on ℓ^2 s.t. $\|T\| < \infty$, where
$$\|T\| = \sup_{\xi \in \ell^2} \frac{\|T\xi\|}{\|\xi\|}$$

Then, $\forall\, m, n \geq 0$ s.t. $(m,n) \neq (0,0)$

$$\varphi_0(S^m (S^*)^n) = \langle S^m (S^*)^n \xi_0, \xi_0 \rangle$$
$$= \langle (S^*)^n \xi_0, (S^*)^m \xi_0 \rangle$$
one of m, $n > 0$
$$= 0$$

Hence, we have a "realization" of our abstract example in a concrete setting.

$*$-Moments of non-unitary isometries & Dyck paths

Def. A path in $\mathbb{Z}^2$ which starts at $(0,0)$ and make $(1, \pm 1)$ steps is called a NE-SE path.

A Dyck path is an NE-SE path which ends on the x-axis and never go strictly below it.

A Dyck path of length k corresponds to $(\lambda_1, \dots, \lambda_k) \in \{\pm 1\}^k$ s.t.

$$\begin{cases} \lambda_1 + \dots + \lambda_j \geq 0 & \forall\, 1 \leq j < k \\ \lambda_1 + \dots + \lambda_k = 0 \end{cases}$$

Proposition. $\forall n \geq 0$, the number of Dyck paths of length $2n$ is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Prop. Let $k>0$ & $\varepsilon(1),\dots,\varepsilon(k)\in\{*,1\}$. Consider the monomial $a^{\varepsilon(1)}\cdots a^{\varepsilon(k)}$.

Set $\lambda_j=\begin{cases}1 & \varepsilon(j)=*\\ -1 & \varepsilon(j)=1\end{cases}$ and γ the NE-SE path corresponding to λ. Then,

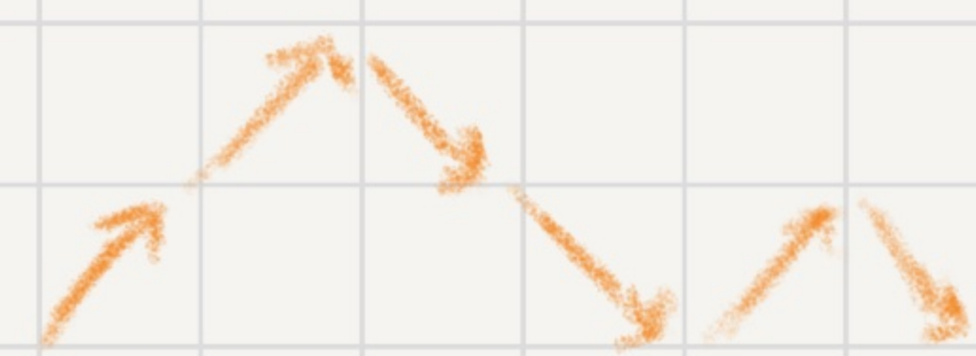
$$\varphi(a^{\varepsilon(1)}\cdots a^{\varepsilon(k)})=\begin{cases}1 & \gamma \text{ is a Dyck path}\\ 0 & \text{o.w.}\end{cases}$$

Examples.

$\varphi(a^*a)=\varphi(1)=1$

$\varphi(aa^*)=0$

$\varphi(a^*a^*aaa^*a)=1$



$*$-distribution of

non unitary isometry

$$a + a^*$$

in the analytic sense

What can we say about the self adjoint element $a+a^*$?

Obs. $$\varphi((a+a^*)^k) = \begin{cases} 0 & k \text{ odd} \\ C_{k/2} & k \text{ even} \end{cases}$$

pf. $$\varphi((a+a^*)^k) = \sum_{\varepsilon(1)\dots\varepsilon(k)\in\{1,*\}} \varphi(a^{\varepsilon(1)}\cdots a^{\varepsilon(k)}) = \sum_{\substack{\text{Dyck paths}\\ \text{of length } k}} 1 .$$

What about the analytic dist ?

Prop. The distribution of $a+a^*$ is given by

$$d\mu(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,dt \qquad \text{in } [-2,2].$$

pf. We want to prove

odd is easy

$$\int_{\mathbb{R}} t^k d\mu(t) = \frac{1}{2\pi}\int_{-2}^{2} t^k \sqrt{4-t^2}\,dt = \frac{1}{p+1}\binom{2p}{p} \qquad k=2p$$

change of variable: $t = 2\cos\theta \implies dt = -2\sin\theta \, d\theta$

$$t \in [-2,2] \implies \theta \in [\pi, 0]$$

$$\int_{-2}^{2} t^{2p} \underbrace{\sqrt{4-t^2}}_{\substack{4-4\cos^2\theta \\ \shortparallel \\ 4\sin^2\theta}} \overbrace{dt}^{-2\sin\theta d\theta} = \int_0^{\pi} 2^{2p+2} \cos^{2p}\theta \, \sin^2\theta \, d\theta = 4^{p+1}\left(I_p - I_{p+1}\right)$$

where

$$I_p = \int_0^{\pi} \cos^{2p}\theta \, d\theta \qquad (p \geq 0)$$

$$= \frac{\pi}{4^p}\binom{2p}{p}$$

with some work

Semi- circular

elements

Def. Let $(\mathcal{A},\varphi)$ be a $*$-ps. Let $a \in \mathcal{A}$ be self adjoint and let $r > 0$. If a has analytic dist

no need to say $*$-...

$$\frac{2}{\pi r^2}\sqrt{r^2 - t^2}\,dt$$

in $[-r, r]$ then we say a is a semicircular element of radius r.

Remark.

$*$ $Var(a) = \frac{r^2}{4}$ and so $r = 2$ is called a standard semicircular.

The Cauchy Transform

&

Stieltjes inversion formula

It was fairly easy to verify that $d\mu(t) = \frac{1}{2\pi}\sqrt{4-t^2}$ is the analytic distribution of $a+a^*$ but how can we find it ourselves?

Def. Let μ be a probability measure on $\mathbb{R}$. The Cauchy transform of μ is the function

$$G_\mu : \mathbb{C}^+ \longrightarrow \mathbb{C}$$
$$z \longmapsto \int \frac{1}{z-t}\, d\mu(t)$$

Lebesgue integrable!

Theorem

* $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ (easy)
* G_μ is analytic on $\mathbb{C}^+$ (recitation)

(obvious)

$G_\mu \dashrightarrow \mu$

$\mu \longleftarrow G_\mu$

harder to work with

fun & insightful

Theorem (Stieltjes Inversion Formula).

Informally: Any probability measure μ on $\mathbb{R}$ can be recovered from G_μ:

For all $a<b$ reals,

$$-\frac{1}{\pi}\cdot\lim_{\varepsilon\searrow 0}\int_a^b \operatorname{Im} G_\mu(x+i\varepsilon)\,dx = \mu((a,b)) + \frac{1}{2}\mu(\{a,b\}).$$

In particular $G_\mu = G_\nu \implies \mu=\nu$.

proof

$$\frac{1}{x-t+i\varepsilon}\,\frac{x-t-i\varepsilon}{x-t-i\varepsilon} = \frac{x-t-i\varepsilon}{(x-t)^2+\varepsilon^2}$$

$$\operatorname{Im} G_\mu(x+i\varepsilon) = \int_{\mathbb{R}} \operatorname{Im}\left(\overbrace{\frac{1}{x-t+i\varepsilon}}\right)d\mu(t)$$

$$= -\int_{\mathbb{R}} \frac{\varepsilon}{(x-t)^2+\varepsilon^2}\,d\mu(t)$$

Thus,

can switch order of integrals due to positivity

Check out Fubini's Theorem

$$\int_a^b \operatorname{Im} G_\mu(x+i\varepsilon)\,dx = -\int_{\mathbb{R}}\left(\int_a^b \frac{\varepsilon}{(x-t)^2+\varepsilon^2}\,dx\right)d\mu(t)$$

$$y = \frac{x-t}{\varepsilon}$$

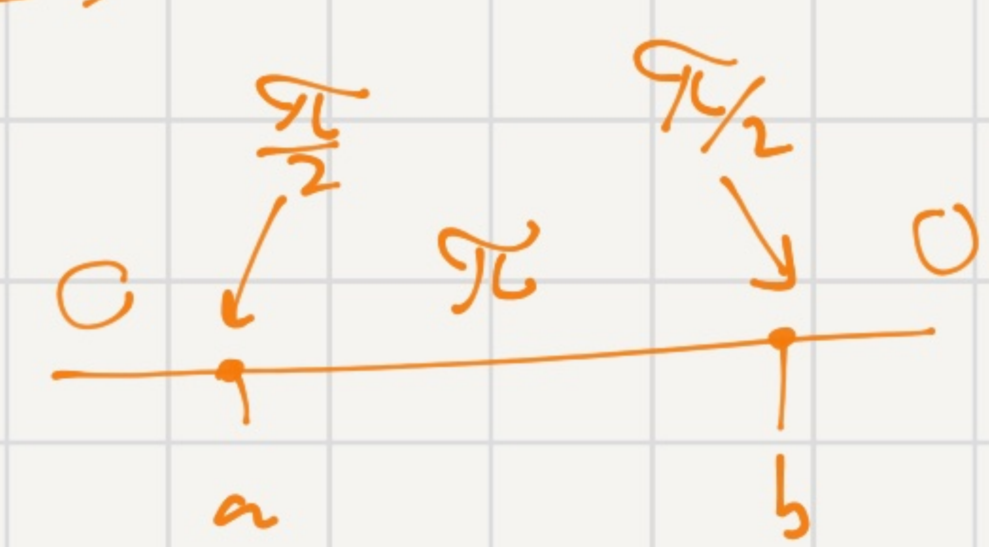
$$\Downarrow$$

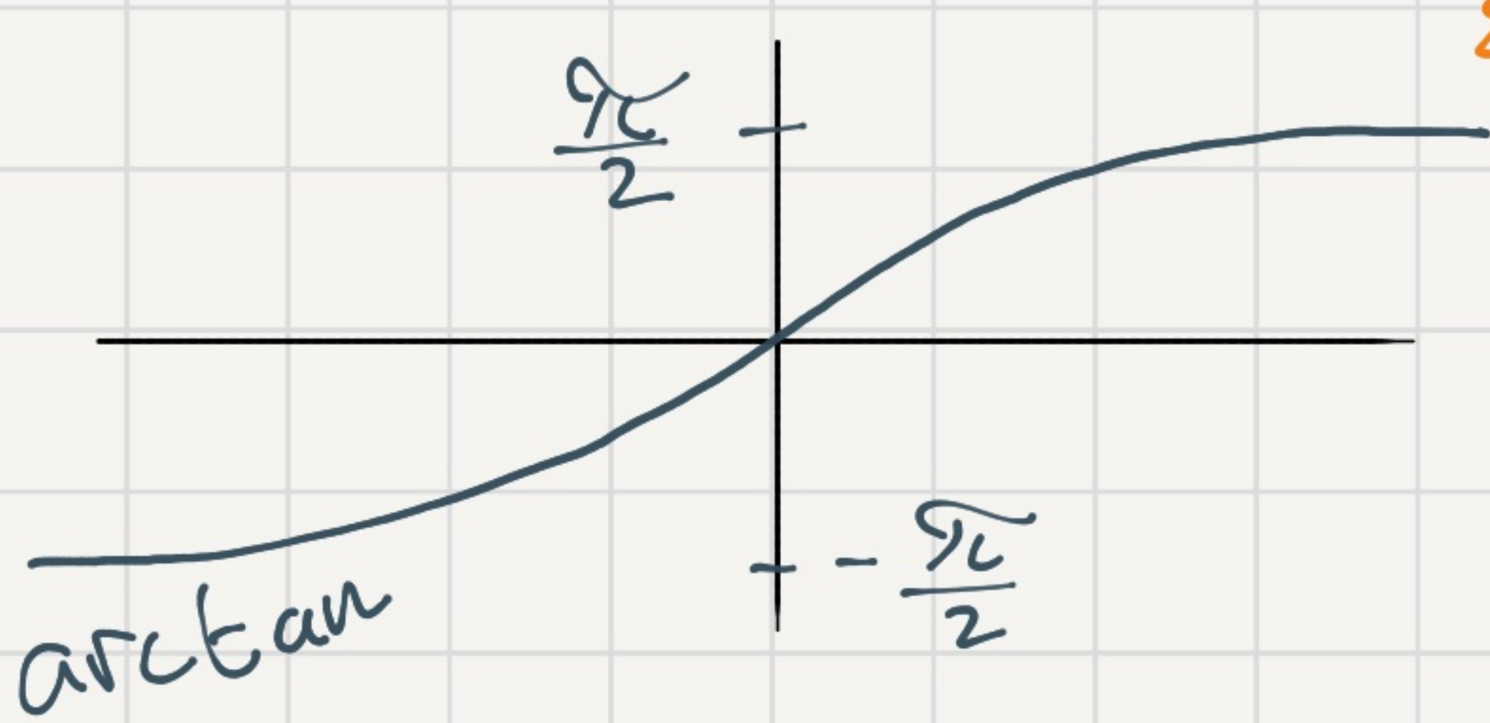
$$\frac{\varepsilon}{(x-t)^2+\varepsilon^2} = \frac{\varepsilon}{(\varepsilon y)^2+\varepsilon^2} = \frac{1}{\varepsilon(y^2+1)}$$

$$\& \ \varepsilon\, dy = dx$$

$$= -\int_{\mathbb{R}}\left(\underbrace{\int_{\frac{a-t}{\varepsilon}}^{\frac{b-t}{\varepsilon}} \frac{1}{1+y^2}\,dy}_{\arctan\left(\frac{b-t}{\varepsilon}\right)-\arctan\left(\frac{a-t}{\varepsilon}\right)}\right)d\mu(t)$$

$$\xrightarrow[\varepsilon \to 0]{}$$





Hence,

$$\lim_{\varepsilon \searrow 0} \int_a^b \operatorname{Im} G_\mu(x+i\varepsilon)\,dx = -\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \arctan\left(\frac{b-t}{\varepsilon}\right) - \arctan\left(\frac{a-t}{\varepsilon}\right) dt$$

Set $f_n = \arctan\left((b-t)n\right) - \arctan\left((a-t)n\right)$

Each f_n is measurable (being continuous).

Now $f := \lim_{n \to \infty} f_n =$

$\in S^+$, hence measurable.

$g(x) := 2$ is integrable & dominates $\forall f_n$ & f $\Longrightarrow$

we can invoke the theorem of dominant convergence.

Hence,

$$\lim_{\varepsilon \to 0} \int_a^b \operatorname{Im} G_\mu(x+i\varepsilon)\,dx = -\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \arctan\left(\frac{b-t}{\varepsilon}\right) - \arctan\left(\frac{a-t}{\varepsilon}\right) d\mu(t)$$

$$= -\int_{\mathbb{R}} \lim_{\varepsilon \to 0} \left(\arctan\left(\frac{b-t}{\varepsilon}\right) - \arctan\left(\frac{a-t}{\varepsilon}\right)\right) d\mu(t)$$

$$= -\int_{\mathbb{R}}$$

$$d\mu(t)$$

$$= -\left[\pi \mu((a,b)) + \frac{\pi}{2} \mu(\{a,b\})\right]$$

Remark. one can remove the $\mu(\{a,b\})$ contribution, hence completely recover μ on open intervals by considering a sequence of open intervals, approaching (a,b) that avoid all the countably many atoms of μ.

closed & bounded

Theorem. Suppose μ is compactly supported. In particular, let $r>0$ be s.t. $\mu((-\infty,-r)\cup(r,\infty))=0$. Then, we have the power series expansion:

$$G_\mu(z)=\sum_{n=0}^{\infty}\frac{m_n}{z^{n+1}} \qquad |z|>r$$

where

$$m_n=\int t^n\,d\mu(t) \quad \text{is the } n^{\text{th}} \text{ moment of } \mu.$$

pf. $\forall z\in\mathbb{C},\ |z|>r,\quad |t|\le r$

$$\frac{1}{z-t}=\frac{1}{z}\,\frac{1}{1-t/z}=\frac{1}{z}\sum_{n=0}^{\infty}\frac{t^n}{z^n}=\sum_{n=0}^{\infty}\frac{t^n}{z^{n+1}}.$$

Hence, $\forall z \in \mathbb{C}$, $|z| > r$,

$$G_\mu(z) = \int \frac{1}{z-t}\, d\mu(t) = \int \sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}}\, d\mu(t)$$

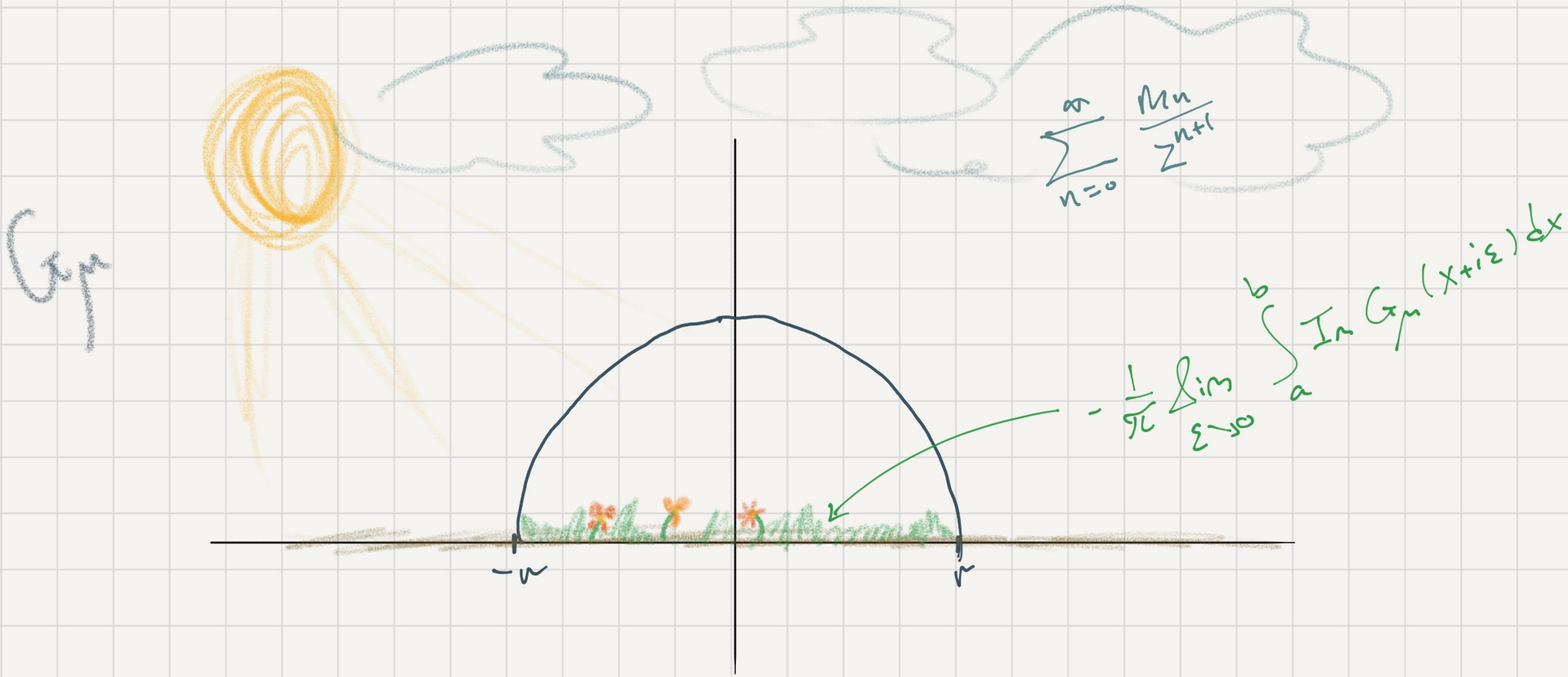
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \underbrace{\int t^n\, d\mu(t)}_{m_n}$$

Corollary from theorem of monotone convergence

Note that our functions are complex-valued...

∎

Stieltjes inversion formula & the above power series expansion "at ∞" in terms of the moments is something I like to call the <u>heaven & earth theorem</u> ☺



Corollary.

$$\lim_{\substack{|z|\to\infty \\ z\in\mathbb{C}^+}} z\,G_\mu(z) = 1$$

pf.

$$z\,G_\mu(z) \underset{\text{for } |z| \text{ large enough}}{=} \underbrace{m_0}_{\int d\mu(t)=1} + \frac{m_1}{z} + \frac{m_2}{z^2} + \cdots \xrightarrow[|z|\to\infty]{} 1$$

∎

Back to our example

Lemma Let μ be a prob. measure with compact support on $\mathbb{R}$

s.t.

$$\int t^k \, d\mu(t) = \begin{cases} 0 & k \text{ odd} \\ C_p = \frac{1}{p+1}\binom{2p}{p} & k = 2p \end{cases}$$

Then,

$$G_\mu(z) = \frac{z - \sqrt{z^2-4}}{2}$$

Pf. For $|z|$ large enough,

$$G_\mu(z) = \sum_{p=0}^{\infty} \frac{C_p}{z^{2p+1}} = \frac{1}{z} + \sum_{p=1}^{\infty} \frac{1}{z^{2p+1}} \overbrace{\left(\sum_{j=1}^{p} C_{j-1} C_{p-j} \right)}^{C_p \quad p \geq 1}$$

$$= \frac{1}{z} + \frac{1}{z} \sum_{p=1}^{\infty} \sum_{j=1}^{p} \frac{C_{j-1}}{z^{2j-1}} \frac{C_{p-j}}{z^{2(p-j)+1}} =$$

$$= \frac{1}{z} + \frac{1}{z} \sum_{j=1}^{\infty} \frac{C_{j-1}}{z^{2j-1}} \underbrace{\sum_{p=j}^{\infty} \frac{C_{p-j}}{z^{2(p-j)+1}}}_{\sum_{k=0}^{\infty} \frac{C_k}{z^{2k+1}} = G_\mu(z)} =$$

$$\frac{1}{z} + \frac{1}{z} \cdot G_\mu(z) \cdot \underbrace{\sum_{j=1}^{\infty} \frac{C_{j-1}}{z^{2j-1}}}_{G_\mu(z)} =$$

$$\frac{1}{z} + \frac{G_\mu(z)^2}{z}$$

So $$G_\mu(z) = \frac{1}{z} + \frac{1}{z} G_\mu(z)^2$$

Thus, $G_\mu(z)$ satisfies the quadratic equation

$$X^2 - zX + 1 = 0$$

(over $\mathbb{C}[[z]]$ - the ring of formal power series)

yes! $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ is still valid

whose solutions are
$$\frac{z \pm \sqrt{z^2-4}}{2}$$

Indeed in $\mathbb{C}[[z]]$!

As $\lim\limits_{\substack{|z|\to\infty \\ z \in \mathbb{C}^+}} z\, G_\mu(z) = 1$, we have that $x(z) = \dfrac{z - \sqrt{z^2-4}}{2}$

$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^2}{10} + \dots$

is an analytic continuation of $G_\mu(z)$ to $\mathbb{C}^+$.

Indeed, $x(z)$ is analytic on
$\mathbb{C} \setminus \{\pm 2 - it \mid t \geq 0\}$

∎

Recap:

$$\text{moments} \xrightarrow{\text{Above Lemma}} G_\mu \xrightarrow{\text{Next}} \mu$$

C_{ap} $\qquad \dfrac{z-\sqrt{z^2-4}}{2}$

For using Stieltjes inversion formula, we wish to calculate

$$\lim_{\varepsilon \searrow 0} \int_a^b \operatorname{Im} G_\mu (x+i\varepsilon)\,dx = \lim_{\varepsilon \searrow 0} \int_a^b \operatorname{Im} \overbrace{\frac{x+i\varepsilon - \sqrt{(x+i\varepsilon)^2-4}}{2}}^{f(x,\varepsilon)}\,dx$$

$$\overset{\text{continuity}}{=} \int_a^b \operatorname{Im} \lim_{\varepsilon \searrow 0} \frac{x+i\varepsilon - \sqrt{(x+i\varepsilon)^2-4}}{2}\,dx$$

$$= \int_a^b \operatorname{Im} \frac{x-\sqrt{x^2-4}}{2}\,dx = (*)$$

$$\left| \int_a^b f(x,\varepsilon+\delta)\,dx - \int_a^b f(x,\varepsilon)\,dx \right| \leq \int_a^b \underbrace{|f(x,\varepsilon+\delta)-f(x,\varepsilon)|}_{\xrightarrow[\delta\to 0]{} 0 \text{ as } f(x,\cdot) \text{ is cont}}\,dx \xrightarrow[\delta \to 0]{} 0$$

as $|[a,b]| < \infty$

$\Longrightarrow$ $\operatorname{supp}(\mu) \subseteq [-2,2]$ & for $-2 \leq a \leq b \leq 2$ we have that

$$(*) = -\int_a^b \frac{\sqrt{4-x^2}}{2}\,dx$$

Thus, by Stieltjes inversion formula,

$$\frac{1}{\pi}\int_a^b \frac{\sqrt{4-x^2}}{2}\,dx = \mu((a,b)) + \underbrace{\cancel{\frac{1}{2}\mu(\{a,b\})}}_{=0}$$

$$\Longrightarrow \quad d\mu(x) = \frac{\sqrt{4-x^2}}{2\pi}\,dx \quad \text{in } [-2,2].$$

Summary.

$$\text{moments} \xrightarrow[\text{Lemma}]{\text{Above}} G_\mu \xrightarrow[\text{inv. formula}]{\text{Stieltjes}} \mu$$

$$C_{ap} \qquad \frac{z-\sqrt{z^2-4}}{2} \qquad d\mu(x) = \frac{\sqrt{4-x^2}}{2\pi}\,dx$$