Free Probability and Ramanujan Graphs - HW #1

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Exercises with * will be graded. Submission in singles or pairs. Contact Gal for questions and clarifications.

1. Prove that $(M_d(\mathbb{C}), \mathsf{tr})$ is a $*$-probability space, where $\mathsf{tr}(A) = \frac{1}{d}\mathsf{Tr}(A)$. Prove that tr is positive and faithful.

2. * Let G be a group, and let $\mathbb{C}G$ be the group algebra:

$$\mathbb{C}G = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{C}, \text{ only finitely many } \alpha_g \neq 0 \right\},$$

with

$$\left(\sum_{g \in G} \alpha_g g\right) \cdot \left(\sum_{h \in G} \beta_h h\right) = \sum_{g,h \in G} \alpha_g \beta_h \cdot gh = \sum_{k \in G} \left(\sum_{g,h : gh = k} \alpha_g \beta_h\right) k,$$

and

$$\left(\sum_{g \in G} \alpha_g g\right)^* = \sum_{g \in G} \bar{\alpha}_g g^{-1}.$$

Let e be the unit of G, and let $\tau_G : \mathbb{C}G \to \mathbb{C}$ be the function defined by

$$\tau_G \left(\sum_{g \in G} \alpha_g g\right) = \alpha_e.$$

(a) Prove that $(\mathbb{C}G, \tau_G)$ is a $*$-probability space. Prove that τ_G is positive and faithful.

(b) Identify the Haar unitaries and p-Haar unitaries of $(\mathbb{C}G, \tau_G)$.

3. * Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. In this exercise we will prove that φ is selfadjoint, that is:

$$\forall a \in \mathcal{A},\ \varphi(a^*) = \overline{\varphi(a)}.$$

(a) Show that if $x \in \mathcal{A}$ is selfadjoint then $x = a^*a - b^*b$ for some $a, b \in \mathcal{A}$.

(b) Deduce that if $x \in \mathcal{A}$ is selfadjoint then $\varphi(x) \in \mathbb{R}$.

(c) Prove that every $a \in \mathcal{A}$ can be expressed uniquely as $a = x + iy$ where x, y are selfadjoint.
Conclude that φ is selfadjoint.

4. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. In this exercise we will prove the Cauchy-Schwarz inequality for $*$-probability spaces:

$$\forall a, b \in \mathcal{A},\ |\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$

   (a) Prove that

   $$\left(\Re\left(\varphi\left(b^*a\right)\right)\right)^2 \leq \varphi(a^*a)\varphi(b^*b)$$

   where $\Re(z)$ is the real part of the complex number z. Begin by examining the function $q : \mathbb{R} \to \mathbb{C}$ defined by

   $$q(t) = \varphi\left((a - tb)^*(a - tb)\right).$$

   (b) Prove the Cauchy-Schwarz inequality, by similarly examining

   $$s(t) = \varphi\left((a - t\alpha b)^*(a - t\alpha b)\right)$$

   for an appropriately chosen $\alpha \in \mathbb{C}$.

5. * Let $p \in \mathbb{N}$. What is the $*$-distribution of a p-Haar unitary s?